ROMANIAN MATHEMATICAL MAGAZINE

UP.621 Find the curvilinear integral of the first kind:

$$\Omega = \int_{\gamma} (x + 2y + z)\,ds;\gamma = [AB];\qquad A(1,2,3);B(3,4,5)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$AB:\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}$$

$$AB:\frac{x-1}{3-1} = \frac{y-2}{4-2} = \frac{z-3}{5-3} = t; t \in [0,1]$$

$$AB:\begin{cases} x - 1 = 2t \\ y - 2 = 2t \\ z - 3 = 2t \end{cases} \Rightarrow AB:\begin{cases} x = 1 + 2t \\ y = 2 + 2t \\ z = 3 + 3t \end{cases}$$

$$\begin{cases} x'(t) = (1+2t)' = 2 \\ y'(t) = (2+2t)' = 2 \\ z'(t) = (3+2t)' = 2 \end{cases}$$

$$ds = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}dt = \sqrt{4+4+4}dt = 2\sqrt{3}dt$$

$$\Omega = \int_{\gamma} (x + 2y + z)\,ds =$$

$$= \int_0^1 (1 + 2t + 2(2+2t) + 3 + 2t) \cdot 2\sqrt{3}dt = 2\sqrt{3}\int_0^1 (8 + 8t)\,dt =$$

$$= 16\sqrt{3}\left(t|_0^1 + \frac{t^2}{2}\Big|_0^1\right) = 16\sqrt{3} \cdot \frac{3}{2} = 24\sqrt{3}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

To parametrize the segment from point $A(1,2,3)$ to point $B(3,4,5)$, we use the vector equation of a line

$$r(t) = A + t(B - A),\quad t \in [0,1]$$

or in coordinates

$$x(t) = 1 + (3-1)t = 1 + 2t$$

$$y(t) = 2 + (4-2)t = 2 + 2t$$

$$z(t) = 3 + (5-3)t = 3 + 2t$$

Next, we find the derivatives of each coordinate with respect to t and compute ds. We have $x'(t) = 2, y'(t) = 2, z'(t) = 2$, and the arc length differential ds is given by

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt$$

$$ds = \sqrt{2^2 + 2^2 + 2^2}dt = \sqrt{4 + 4 + 4}dt = \sqrt{12}dt = 2\sqrt{3}dt$$

Substituting the parameterizations and ds into the integral over the interval $t \in [0, 1]$ yields

$$\Omega = \int_0^1 [(1 + 2t) + 2(2 + 2t) + (3 + 2t)] \cdot 2\sqrt{3}dt = 16\sqrt{3}\int_0^1 (1 + t)\, dt$$

Now, integrate with respect to t to obtain

$$\Omega = 16\sqrt{3}\left[t + \frac{t^2}{2}\right]_0^1 = 16\sqrt{3}\left(1 + \frac{1}{2}\right) = 16\sqrt{3} \cdot \frac{3}{2} = 24\sqrt{3}$$

Thus,

$$\Omega = 24\sqrt{3}$$