ROMANIAN MATHEMATICAL MAGAZINE

UP.617 Calculate the integral:

$$\int_{-1}^{1} \frac{x^2 \arccos(x)}{\sqrt{1-x^6}} dx$$

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Solution 1 by proposer

Let us denote:

$$A = \int_{-1}^{1} \frac{x^2 \arccos(x)}{\sqrt{1-x^6}} dx; B = \int_{-1}^{1} \frac{x^2 \arcsin(x)}{\sqrt{1-x^6}} dx$$

We have:

$$A + B = \frac{\pi}{2} \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx$$

But:

$$B = \int_{-1}^{1} \frac{x^2 \arcsin(x)}{\sqrt{1-x^6}} dx = 0$$

because the function to be integrate is odd.

Let us denote:

$$C = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^6}} dx$$

We have:

$C = 2D$, **where:** $D = \int_0^1 \frac{x^2}{\sqrt{1-x^6}} dx$, **because the function to be integrated is even.**

We obtain:

$$A = \pi D$$

We consider the integral D.

$$D = \int_{0}^{1} \frac{x^2}{\sqrt{1-x^6}} dx$$

In the integral D **we make the variable change** $x^6 = y$

We obtain:

$$D=\frac{1}{6}\int_0^1 y^{-\frac{1}{2}}(1-y)^{-\frac{1}{2}}dy$$

Let us denote:

$$E=\int_0^1 y^{-\frac{1}{2}}(1-y)^{-\frac{1}{2}}dy$$

So:

$$D=\frac{1}{6}E$$

We will use the Euler's Beta function:

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt$$

Result:

$$E=B\left(\frac{1}{2},\frac{1}{2}\right),\qquad E=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}=\pi$$

Where $\Gamma(\alpha)$ is the Euler's Gamma function

$$D=\frac{1}{6}E=\frac{1}{6}\pi$$

$$A=\pi D=\frac{\pi^2}{6}$$

We obtained the value of the integral A:

$$A=\frac{\pi^2}{6}$$

Thus, the problem is solved.

Solution 2 by Jose Luis Diaz Barrero-Spain

$$g(x)=\frac{x^2}{\sqrt{1-x^6}},$$

which is an even function on $[-1,1]$. Split the integral and substitute $x\to -x$ in the piece over $[-1,0]$, to obtain

$$I=\int_0^1 g(x)\arccos(-x)\,dx+\int_0^1 g(x)\arccos(x)\,dx=$$

$$= \int_0^1 g(x)\,[\arccos(x) + \arccos(-x)]dx.$$

Using the identity

$$\arccos(x) + \arccos(-x) = \pi, \quad x \in [-1, 1],$$

this simplifies to

$$I = \pi \int_0^1 g(x)\,dx = \pi \int_0^1 \frac{x^2}{\sqrt{1 - x^6}}dx.$$

Let $u = x^3$, so $du = 3x^2dx$. When $x = 0, u = 0$ and when $x = 1, u = 1$. Then

$$\int_0^1 \frac{x^2}{\sqrt{1 - x^6}}dx = \frac{1}{3}\int_0^1 \frac{du}{\sqrt{1 - u^2}} = \frac{1}{3}[\arcsin(u)]_0^1 = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6},$$

and

$$I = \pi \cdot \frac{\pi}{6} = \frac{\pi^2}{6}.$$