UP.616 Calculate the following integral:

$$\int_0^\infty \frac{\ln x}{(x+1)(x^4+x^2+1)}dx$$

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Solution 1 by proposer

Let us denote:

$$I = \int_0^\infty \frac{\ln x}{(x+1)(x^4+x^2+1)}dx$$

$$A = \int_0^1 \frac{\ln x}{(x+1)(x^4+x^2+1)}dx = \int_0^1 \frac{(1-x)\ln x}{1-x^6}dx$$

$$B = \int_1^\infty \frac{\ln x}{(x+1)(x^4+x^2+1)}dx = \int_1^\infty \frac{(1-x)\ln x}{1-x^6}dx$$

We consider the integral A.

We have:

$$A = \int_0^1 \frac{\ln x}{1-x^6}dx - \int_0^1 \frac{x\ln x}{1-x^6}dx$$

We will calculate:

$$A_1 = \int_0^1 \frac{\ln x}{1-x^6}dx$$

We make the variable change: $x^6 = y; x = y^{\frac{1}{6}}$.

We obtain:

$$A_1 = \frac{1}{36}\int_0^1 \frac{y^{-\frac{5}{6}}\ln y}{1-y}dy$$

We will use the following known relationship:

$\int_0^1 \frac{z^a \ln z}{1-z}dz = -\Psi(a+1), a \in \mathbb{R} \setminus \{-1,-2,\dots\}$, where $\Psi_1(x)$ is the trigamma function

We obtain:

$$A_1 = -\frac{1}{36}\Psi_1\left(\frac{1}{6}\right)$$

We will calculate:

$$A_2 = \int_0^1 \frac{x\ln x}{1-x^6}dx$$

Proceeding similarly, we obtain:

$$A_2 = -\frac{1}{36}\Psi\left(\frac{2}{6}\right)$$

We obtained the value of the integral A:

$$A = A_1 - A_2 = \frac{1}{36}\left[-\Psi_1\left(\frac{1}{6}\right) + \Psi\left(\frac{2}{6}\right)\right]$$

We consider the integral B. We make the variable change: $x = \frac{1}{t}$.

We obtain:

$$B = \int_0^1 \frac{(t-1)t^3 \ln t}{1-t^6} dt$$

By proceeding similarly to the integral A, we obtain:

$$B = \frac{1}{36}\left[\Psi_1\left(\frac{4}{6}\right) - \Psi_1\left(\frac{5}{6}\right)\right]$$

Result:

$$I = A + B$$

$$I = \frac{1}{36}\left[-\Psi_1\left(\frac{1}{6}\right) + \Psi\left(\frac{2}{6}\right) + \Psi_1\left(\frac{4}{6}\right) - \Psi_1\left(\frac{5}{6}\right)\right]$$

We use the reflection formula:

$$\Psi_{1(x)} + \Psi_1(1-x) = \frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\Psi_1\left(\frac{1}{6}\right) + \Psi_1\left(\frac{5}{6}\right) = \frac{\pi^2}{\sin^2\frac{\pi}{6}} = 4\pi^2$$

$$\Psi\left(\frac{2}{6}\right) + \Psi_1\left(\frac{4}{6}\right) = \frac{\pi^2}{\sin^2\frac{\pi}{3}} = \frac{4\pi^2}{3}$$

Result:

$$I = \frac{1}{36}\left[-4\pi^2 + \frac{4\pi^2}{3}\right], \qquad I = -\frac{2\pi^2}{27}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

Putting $x = \frac{1}{t}$, we have $dx = -\frac{dt}{t^2}$, $\ln x = -\ln t$, $(x+1)(x^4+x^2+1) = \frac{(1+t)(1+t^2+t^4)}{t^5}$, and

$$I = -\int_0^\infty \frac{t^3 \ln t}{(t+1)(t^4+t^2+1)} dt.$$

Define

$$A = \int_0^\infty \frac{\ln x}{(x+1)(x^4+x^2+1)} dx = I, \qquad B = \int_0^\infty \frac{x^3 \ln x}{(x+1)(x^4+x^2+1)} dx,$$

such that $I = -B$. Adding up the two representations, yields

$$A - B = 2I = \int_0^\infty \frac{(1-x^3)\ln x}{(x+1)(x^4+x^2+1)} dx$$

Using $1 - x^3 = (1-x)(1+x+x^2)$ and the factorization of $x^4 + x^2 + 1$, we have

$$\frac{1-x^3}{(x+1)(x^4+x^2+1)} = \frac{(1-x)(x^2+x+1)}{(x+1)(x^2+x+1)(x^2-x+1)}$$
$$= \frac{1-x}{(x+1)(x^2-x+1)} = \frac{1-x}{x^3+1}.$$

Therefore,

$$2I = \int_0^\infty \frac{(1-x)\ln x}{x^3+1} dx = \int_0^\infty \frac{\ln x}{x^3+1} dx - \int_0^\infty \frac{x \ln x}{x^3+1} dx.$$

Using the well – known standard integral

$$\int_0^\infty \frac{x^{s-1}}{1+x^n} dx = \frac{\pi}{n \sin\left(\frac{\pi s}{n}\right)}, \quad 0 < s < n,$$

and computing its derivative respect to s yields

$$\int_0^\infty \frac{x^{s-1}\ln x}{1+x^n} dx = -\frac{\pi^2}{n^2} \frac{\cos\left(\frac{\pi s}{n}\right)}{\sin^2\left(\frac{\pi s}{n}\right)}.$$

In particular, for Para $n = 3, s = 1$, we obtain

$$\int_0^\infty \frac{\ln x}{1+x^3} dx = -\frac{\pi^2}{9} \cdot \frac{\cos\left(\frac{\pi}{3}\right)}{\sin^2\left(\frac{\pi}{3}\right)} = -\frac{\pi^2}{9} \cdot \frac{\frac{1}{2}}{\frac{3}{4}} = -\frac{2\pi^2}{27}.$$

For $n = 3, s = 2$, we obtain

$$\int_0^\infty \frac{x \ln x}{1+x^3} dx = -\frac{\pi^2}{9} \cdot \frac{\cos\left(\frac{2\pi}{3}\right)}{\sin^2\left(\frac{2\pi}{3}\right)} = -\frac{\pi^2}{9} \cdot \frac{-\frac{1}{2}}{\frac{3}{4}} = \frac{2\pi^2}{27}$$

Then,

$$2I = \left(-\frac{2\pi^2}{27}\right) - \left(\frac{2\pi^2}{27}\right) = -\frac{4\pi^2}{27}.$$

and

$$I = \int_0^\infty \frac{\ln x}{(x+1)(x^4+x^2+1)} dx = -\frac{2\pi^2}{27}$$