SP. 630 In ΔABC the following relationship holds:

$$\frac{a^2}{h_b^3}+\frac{b^2}{h_c^3}+\frac{c^2}{h_a^3}\geq\frac{8}{3R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\frac{a^2}{h_b^3}+\frac{b^2}{h_c^3}+\frac{c^2}{h_a^3}\geq 3\sqrt[3]{\frac{(abc)^2}{(h_a h_b h_c)^3}}=$$

$$=\frac{3}{h_a h_b h_c}\cdot\sqrt[3]{(4Rrs)^2}\overset{EULER}{\geq}\frac{3}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}\cdot\sqrt[3]{(4\cdot 2r\cdot rs)^2}=$$

$$=\frac{3abc}{8F^3}\cdot\sqrt[3]{(8r^2s)^2}\overset{MITRINOVIC}{\geq}\frac{3\cdot 4RF}{8F^3}\cdot\sqrt[3]{\left(8r^2\cdot 3\sqrt{3}r\right)^2}=\frac{3R}{2F^2}\cdot\sqrt[3]{64\cdot 27\cdot r^6}=$$

$$=\frac{3R}{2r^2s^2}\cdot 4\cdot 3\cdot r^2=\frac{9\cdot 2\cdot R}{s^2}\overset{MITRINOVIC}{\geq}\frac{18R}{\frac{27R^2}{4}}=\frac{72}{27R}=\frac{8}{3R}$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu– Romania

$$LHS=\sum\frac{a^2}{h_b^3}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{a^2}{h_b^3}}=3\sqrt[3]{\frac{a^2b^2c^2}{h_a^3h_b^3h_c^3}}=3\sqrt[3]{\frac{16R^2r^2p^2}{\left(\frac{2r^2p^2}{R}\right)^3}}=3\sqrt[3]{\frac{2R^5}{r^4p^4}}\overset{Mitrinovic}{\geq}$$

$$\geq 3\sqrt[3]{\frac{2R^5}{r^4\left(\frac{27R^2}{4}\right)^2}}=\frac{2}{3r}\sqrt[3]{\frac{4R}{r}}\overset{Euler}{\geq}\frac{8}{3R}=RHS.$$

We have used above $\frac{2}{3r}\sqrt[3]{\frac{4R}{r}}\overset{Euler}{\geq}\frac{8}{3R}\Leftrightarrow R^4\geq 16r^4$, see $R\geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed. In ΔABC holds:

$$\sum\frac{a^n}{h_b^{n+1}}\geq\frac{2^{n+1}}{3^{\frac{n}{2}}R},n\in\mathbb{N}$$

Marin Chirciu

Solution:

$$LHS=\sum\frac{a^n}{h_b^{n+1}}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{a^n}{h_b^{n+1}}}=3\sqrt[3]{\frac{a^nb^nc^n}{h_a^{n+1}h_b^{n+1}h_c^{n+1}}}=3\sqrt[3]{\frac{4^nR^nr^np^n}{\left(\frac{2r^2p^2}{R}\right)^{n+1}}}=$$

$$=3\sqrt[3]{\frac{2^{n-1}R^{2n+1}}{r^{n+2}p^{n+2}}}\overset{Mitrinovic}{\geq}3\sqrt[3]{\frac{2^{n-1}R^{2n+1}}{r^{n+2}\left(\frac{3R\sqrt{3}}{2}\right)^{n+2}}}=\frac{1}{3^{\frac{n}{2}}}\sqrt[3]{\frac{2^{2n+1}R^{n-1}}{r^{n+2}}}\overset{Euler}{\geq}\frac{2^{n+1}}{3^{\frac{n}{2}}R}=RHS.$$

We have used above $\frac{1}{3^{\frac{n}{2}}}\sqrt[3]{\frac{2^{2n+1}R^{n-1}}{r^{n+2}}}\overset{Euler}{\geq}\frac{2^{n+1}}{3^{\frac{n}{2}}}\Leftrightarrow R^{n+2}\geq(2r)^{n+2}$, see $R\geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Note: For $n=2$ we obtain Problem SP.630 from RMM Number 42 – Autumn 2026.

SP.630. In ΔABC holds:

$$\sum\frac{a^2}{h_b^3}\geq\frac{8}{3R}$$

Nguyen Hung Cuong – Vietnam

Solution 3 by José Luis Díaz – Barrero – Spain

By the Law of Sines, we have $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$. Since $S=\frac{1}{2}ac\sin B$ and $h_b=\frac{2S}{b}$, substituting gives

$$h_a=2R\sin B\sin C,\quad h_b=2R\sin A\sin C,\quad h_c=2R\sin A\sin B$$

Then,

$$\frac{a^2}{h_b^3}=\frac{1}{2R\sin A\sin^3 C},\frac{b^2}{h_c^3}=\frac{1}{2R\sin^3 A\sin B},\frac{c^2}{h_a^3}=\frac{1}{2R\sin^3 B\sin C}$$

Writing $x=\sin A$, $y=\sin B$, $z=\sin C$, the inequality is equivalent to

$$\frac{1}{xz^3}+\frac{1}{x^3y}+\frac{1}{y^3z}\geq\frac{16}{3}.$$

On account of AM-GM inequality, we have

$$\frac{1}{xz^3}+\frac{1}{x^3y}+\frac{1}{y^3z}\geq\left(\frac{1}{xz^3}\cdot\frac{1}{x^3y}\cdot\frac{1}{y^3z}\right)^{\frac{1}{3}}=\frac{3}{(xyz)^{\frac{4}{3}}}.$$

Since $f(t) = \ln(\sin t)$ is concave on $(0, \pi)$, Jensen's inequality gives

$$\sin A \sin B \sin C \leq \sin^3\left(\frac{A+B+C}{3}\right) = \sin^3\frac{\pi}{3} = \frac{3\sqrt{3}}{8},$$

so

$$(xyz)^{\frac{4}{3}} \leq \left(\frac{3\sqrt{3}}{8}\right)^{\frac{4}{3}} = \frac{3^2}{8^{\frac{4}{3}}} = \frac{9}{16} \Rightarrow \frac{3}{(xyz)^{\frac{4}{3}}} \geq \frac{3}{\frac{9}{16}} = \frac{16}{3}$$

Finally, combining the preceding, yields

$$\frac{a^2}{h_b^3} + \frac{b^2}{h_c^3} + \frac{c^2}{h_a^3} = \frac{1}{2R}\left(\frac{1}{xz^3} + \frac{1}{x^3y} + \frac{1}{y^3z}\right) \geq \frac{1}{2R} \cdot \frac{16}{3} = \frac{8}{3R}.$$

Equality holds if and only if $A = B = C = \frac{\pi}{3}$. That is, when ΔABC is equilateral.