ROMANIAN MATHEMATICAL MAGAZINE

SP.629 If $a,b>0;a+b\le 2$ then:

$$\left(\frac{2+a}{1+a}+\frac{1-2b}{1+2b}\right)\left(\frac{2+3a}{1+3a}+\frac{1-6b}{1+6b}\right)>\frac{64}{105}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{2+a}{1+a}+\frac{1-2b}{1+2b}=\left(\frac{1}{1+a}+1\right)+\left(\frac{2}{1+2b}-1\right)=$$

$$=\frac{1}{1+a}+\frac{2}{1+2b}=\frac{2}{2+2a}+\frac{2}{1+2b}=$$

$$=2\left(\frac{1}{2+2a}+\frac{1}{1+2b}\right)\overset{BERGSTROM}{\ge}2\left(\frac{(1+1)^2}{2+2a+1+2b}\right)=$$

$$=2\cdot\frac{4}{3+4b}=\frac{8}{3+2(a+b)}\ge\frac{8}{3+2\cdot 2}=\frac{8}{7}$$

$$\frac{2+a}{1+a}+\frac{1-2b}{1+2b}\ge\frac{8}{7}\quad (1)$$

$$\frac{2+3a}{1+3a}+\frac{1-6b}{1+6b}=\left(\frac{1}{1+3a}+1\right)+\left(\frac{2}{1+6b}-1\right)=$$

$$=\frac{1}{1+3a}+\frac{2}{1+6b}=\frac{2}{2+6a}+\frac{2}{1+6b}=2\left(\frac{1}{2+6a}+\frac{1}{1+6b}\right)\ge$$

$$\overset{BERGSTROM}{\ge}2\cdot\frac{(1+1)^2}{2+6a+1+6b}=\frac{8}{3+6(a+b)}\ge\frac{8}{3+6\cdot 2}=\frac{8}{15}$$

$$\frac{2+3a}{1+a}+\frac{1-2b}{1+2b}\ge\frac{8}{15}\quad (2)$$

By multiplying (1);(2):

$$\left(\frac{2+a}{1+a}+\frac{1-2b}{1+2b}\right)\left(\frac{2+3a}{1+a}+\frac{1-2b}{1+2b}\right)>\frac{8}{7}\cdot\frac{8}{15}=\frac{64}{105}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

If $a,b>0$ and $a+b\le 2$, then prove that

$$\left(\frac{2+a}{1+a}+\frac{1-2b}{1+2b}\right)\left(\frac{2+3a}{1+3a}+\frac{1-6b}{1+6b}\right)\ge\frac{64}{105}$$

$$\frac{2+a}{1+a}=1+\frac{1}{1+a},\qquad \frac{1-2b}{1+2b}=-1+\frac{2}{1+2b},$$

the first bracket becomes

$$\frac{2+a}{1+a}+\frac{1-2b}{1+2b}=\frac{1}{1+a}+\frac{2}{1+2b}.$$

Likewise,

$$\frac{2+3a}{1+3a}+\frac{1-6b}{1+6b}=\frac{1}{1+3a}+\frac{2}{1+6b}.$$

So it suffices to prove, for $a,b>0$ with $a+b\leq 2$,

$$F(a,b):=\left(\frac{1}{1+a}+\frac{2}{1+2b}\right)\left(\frac{1}{1+3a}+\frac{2}{1+6b}\right)\geq\frac{64}{105}.$$

Fix $a>0$. Each bracket of $F(a,b)$ is a positive constant plus a decreasing function of b, hence each bracket is a positive, decreasing function of b. A product of two positive decreasing functions is decreasing, so $b\to F(a,b)$ is decreasing.

Since the constraint gives $0<b\leq 2-a$, decreasingness yields

$F(a,b)\geq F(a,2-a)$ for all admissible a,b.

So it suffices to prove $F(a,2-a)\geq\frac{64}{105}$ for $a\in(0,2)$.

Set $b=2-a$. Combine each bracket into a single fraction

$$\frac{1}{1+a}+\frac{2}{5-2a}=\frac{(5-2a)+2(1+a)}{(1+a)(5-2a)}=\frac{7}{(1+a)(5-2a)},$$

since $(5-2a)+2(1+a)=7$ identically. Similarly,

$$\frac{1}{1+3a}+\frac{2}{13-6a}=\frac{(13-6a)+2(1+3a)}{(1+3a)(13-6a)}=\frac{15}{(1+3a)(13-6a)}.$$

$$F(a,2-a)=\frac{7\cdot 15}{(1+a)(5-2a)(1+3a)(13-6a)}=\frac{105}{A(a)B(a)},$$

where

$$A(a)=(1+a)(5-2a),\ \ B(a)=(1+3a)(13-6a).$$

Both $A(a),B(a)>0$ for $a\in(0,2)$, since $A>0$ on $\left(-1,\frac{5}{2}\right)$ and $B>0$ on $\left(-\frac{1}{3},\frac{13}{6}\right)$.

Next, we bound $A(a)$ and $B(a)$ via AM-GM:

- Bounding A: Note $2(1+a)+(5-2a)=7$ is constant. By AM-GM,

$$2(1+a)(5-2a)\leq\left(\frac{2(1+a)+(5-2a)}{2}\right)^2=\left(\frac{7}{2}\right)^2=\frac{49}{4}\Rightarrow A(a)\leq\frac{49}{8},$$

with equality when $2(1+a)=5-2a$, i.e. $a=\frac{3}{4}$.

- Bounding B: Note $2(1+3a)+(13-6a)=15$ is constant. By AM-GM,

$$2(1+3a)(13-6a)\leq\left(\frac{2(1+3a)+(13-6a)}{2}\right)^2=\left(\frac{15}{2}\right)^2=\frac{225}{4}\Rightarrow B(a)\leq\frac{225}{8},$$

with equality when $2(1+3a)=13-6a$, i.e. $a=\frac{11}{12}$.

Since $A(a),B(a)>0$, multiplying the two bounds gives

$$A(a)B(a)\leq\frac{49}{8}\cdot\frac{225}{8}=\frac{11025}{64}=\frac{105^2}{64}.$$

$$F(a,2-a)=\frac{105}{A(a)B(a)}\geq\frac{105}{\frac{11025}{64}}=\frac{105\cdot 64}{105^2}=\frac{64}{105}.$$

Combining with the preceding, yields

$F(a,b)\geq F(a,2-a)\geq\frac{64}{105}$ for all $a,b>0,a+b\leq 2$.

This is precisely the desired inequality. The two AM-GM steps attain equality at $a=\frac{3}{4}$ and $a=\frac{11}{12}$ respectively, which cannot hold simultaneously, so the bound $\frac{64}{105}$ is never actually attained. The true minimum of the left-hand side is slightly larger (numerically about 0.6132, attained near $a\approx 0.86,b\approx 1.14$). Thus $\frac{64}{105}$ is a clean, provable, though not sharp, lower bound.