

# ROMANIAN MATHEMATICAL MAGAZINE

**SP.628** If  $a, b, c > 0$ ;  $(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 17$  then:

$$(a^2 + b^2 + c^2) \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \geq 73$$

*Proposed by Daniel Sitaru – Romania*

**Solution by proposer**

Denote:  $u = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ ;  $v = \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$

$$17 = (a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \left( \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) + \left( \frac{b}{a} + \frac{a}{c} + \frac{c}{b} \right) + 3 = u + v + 3$$

$$\Rightarrow 17 = u + v + 3 \Rightarrow u + v = 14$$

$$(a^2 + b^2 + c^2) \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) = \left( \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \right) + \left( \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{b^2} \right) + 3 =$$

$$= 3 + (u^2 - 2v) + (v^2 - 2u) = 3 + \left( \frac{u^2}{1} + \frac{v^2}{1} \right) - 2(u + v) \geq$$

$$\stackrel{\text{BERGSTROM}}{\geq} 3 + \frac{(u + v)^2}{1 + 1} - 2(u + v) = 3 + \frac{14^2}{2} - 2 \cdot 14 = 3 + 98 - 28 = 73$$

Equality holds for:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{b}{a} + \frac{a}{c} + \frac{c}{b} = 7$$