ROMANIAN MATHEMATICAL MAGAZINE

SP.627 Solve for reals:

$$\sqrt[3]{(1+x)^2}+\sqrt[3]{(1-x)^2}=\sqrt[3]{1+x}+\sqrt[3]{1-x}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

Denote: $\begin{cases} a=\sqrt[3]{1+x} \\ b=\sqrt[3]{1-x} \end{cases} \Rightarrow a^3+b^3=2$

Denote: $\begin{cases} S=a+b \\ P=ab \end{cases} \Rightarrow \begin{cases} S^3-3SP=2 \\ S^2-2P=S \end{cases}$

$$2P=S^2-S \Rightarrow P=\frac{S^2-S}{2}$$

$$S^3-3S\cdot\frac{S^2-S}{2}=2 \Rightarrow 2S^3-3S^3+3S^2=4$$

$$-S^3+3S^2-4=0 \Rightarrow S^3-3S^2+4=0$$

$$S^3+S^2-4S^2+4=0 \Rightarrow S^2(S+1)-4(S^2-1)=0$$

$$S^2(S+1)-4(S-1)(S+1)=0 \Rightarrow (S+1)(S^2-4S+4)=0$$

$$S+1=0 \Rightarrow S=-1 \Rightarrow P=1 \Rightarrow z^2-Sz+P=0 \Rightarrow z^2+z+1=0; \Delta<0$$

$$S^2-4S+4=0 \Rightarrow (S-2)^2=0 \Rightarrow S=2 \Rightarrow P=1$$

$$\Rightarrow a=b=1 \Rightarrow \sqrt[3]{1-x}=\sqrt[3]{1+x}=1 \Rightarrow x=0$$

Solution 2 by Marin Chirciu– Romania

With the substitution $\left(\sqrt[3]{1+x},\sqrt[3]{1-x}\right)=(a,b) \Rightarrow \begin{cases} a^3+b^3=2 \\ a^2+b^2=a+b \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} S^3-3SP=2 \\ S^2-2P=S \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} S^3-3SP=2 \\ P=\dfrac{S^2-S}{2} \end{cases} \Leftrightarrow \begin{cases} S^3-3S\cdot\dfrac{S^2-S}{2}=2 \\ P=\dfrac{S^2-S}{2} \end{cases} \Leftrightarrow \begin{cases} S^3-3S^2-4=0 \\ P=\dfrac{S^2-S}{2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (S-2)^2(S+1)=0 \\ P=\dfrac{S^2-S}{2} \end{cases} \Leftrightarrow (S,P)\in\{(2,1),(-1,1)\}.$$

1. $(S,P)=(2,1)\Leftrightarrow(a,b)=(1,1)\Leftrightarrow\left(\sqrt[3]{1+x},\sqrt[3]{1-x}\right)=(1,1)\Leftrightarrow x=0.$

2. $(S,P)=(-1,1)\Leftrightarrow a^2-Sa+P=0\Leftrightarrow a^2+a+1=0,\Delta=-3<0,a\notin\mathbb{R}$

We deduce that $x=0$ is the unique solution of the equation.