ROMANIAN MATHEMATICAL MAGAZINE

SP.626 If $a,b,c>0, a+b+c=1$ and $\lambda\ge 1$ then:

$$\sum\frac{a+\sqrt{bc}}{a+\lambda}\le\frac{1+3\lambda}{4}+\frac{1}{\lambda+1}\sum\sqrt{bc}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We have:

$$\sum\frac{a+\sqrt{bc}}{a+\lambda}\le\frac{1+3\lambda}{4}+\frac{1}{\lambda+1}\sum\sqrt{bc}\Leftrightarrow\sum\left(\frac{a+\sqrt{bc}}{a+\lambda}-\frac{1}{\lambda+1}\sqrt{bc}\right)\le\frac{1+3\lambda}{4}$$

which follows from:

$$\sum\left(\frac{a+\sqrt{bc}}{a+\lambda}-\frac{1}{\lambda+1}\sqrt{bc}\right)=\sum\frac{(\lambda+1)a+(1-a)\sqrt{bc}}{2(a+1)}\overset{AGM}{\le}$$

$$\le\sum\frac{(\lambda+1)a+(1-a)\frac{b+c}{2}}{2(a+1)}=$$

$$=\sum\frac{2(\lambda+1)a+(1-a)(b+c)}{4(a+\lambda)}=\sum\frac{2(\lambda+1)a+(1-a)(1-a)}{4(a+\lambda)}$$

$$=\sum\frac{a^2+2\lambda a+1}{4(a+\lambda)}$$

$$=\sum\frac{a^2+2\lambda a+\lambda^2+1-\lambda^2}{4(a+\lambda)}=\sum\frac{a^2+2\lambda a+\lambda^2}{4(a+\lambda)}+\sum\frac{1-\lambda^2}{4(a+\lambda)}=$$

$$=\sum\frac{(a+\lambda)^2}{4(a+\lambda)}+\sum\frac{1-\lambda^2}{4(a+\lambda)}=$$

$$=\sum\frac{a+\lambda}{4}+\sum\frac{1-\lambda^2}{4(a+\lambda)}=\frac{\sum a+3\lambda}{4}+\sum\frac{1-\lambda^2}{4(a+\lambda)}=$$

$$=\frac{1+3\lambda}{4}+\sum\frac{1-\lambda^2}{4(a+\lambda)}\overset{\lambda\ge 1}{\ge}\frac{1+3\lambda}{4}$$

Equality holds if and only if $a=b=c=\frac{1}{3}$ and $\lambda=1$.

Note: For $\lambda=1$ we obtain Cyclic Inequality proposed by Phan Ngoc Chau in RMM 2/2022.

If $a,b,c>0, a+b+c=1$ then:

$$\sum\frac{a+\sqrt{bc}}{a+1}\le 1+\frac{1}{2}\sum\sqrt{bc}$$

Phan Ngoc Chau – Vietnam

Solution 2 by Jose Luis Diaz Barrero-Spain

Let $S=\sum_{cyc}\sqrt{bc}$. Since $1-a=b+c$, then so

$$\frac{a+\sqrt{bc}}{a+\lambda}=\frac{a}{a+\lambda}+\frac{\sqrt{bc}}{\lambda+1}+\frac{\sqrt{bc}(b+c)}{(a+\lambda)(\lambda+1)}$$

Multiplying by $\sqrt{bc}$ the inequality

$$\frac{1}{a+\lambda}=\frac{1}{\lambda+1}+\frac{1-a}{(a+\lambda)(\lambda+1)},$$

yields

$$\frac{\sqrt{bc}}{a+\lambda}=\frac{\sqrt{bc}}{\lambda+1}+\frac{\sqrt{bc}(1-a)}{(a+\lambda)(\lambda+1)}.$$

Using the constraint $a+b+c=1$, or $1-a=b+c$, we have

$$\frac{\sqrt{bc}}{a+\lambda}=\frac{\sqrt{bc}}{\lambda+1}+\frac{\sqrt{bc}(b+c)}{(a+\lambda)(\lambda+1)}.$$

Adding $\frac{a}{a+\lambda}$ to both sides, we get

$$\frac{a+\sqrt{bc}}{a+\lambda}=\frac{a}{a+\lambda}+\frac{\sqrt{bc}}{a+\lambda}=\frac{a}{a+\lambda}+\frac{\sqrt{bc}}{\lambda+1}+\frac{\sqrt{bc}(b+c)}{(a+\lambda)(\lambda+1)},$$

which is the desired identity.

Summing cyclically, the middle term contributes $\frac{S}{\lambda+1}$, which cancels with the same term on the right-hand side of the claim. Thus it suffices to prove

$$\sum_{cyc}\frac{a}{a+\lambda}+\frac{1}{\lambda+1}\sum_{cyc}\frac{\sqrt{bc}(b+c)}{a+\lambda}\le\frac{1+3\lambda}{4} \qquad (1)$$

By AM-GM, $\sqrt{bc}\le\frac{b+c}{2}$, hence $\sqrt{bc}(b+c)\le\frac{(1-a)^2}{2}$, so it is enough to show $\sum_{cyc}g(a)\le\frac{1+3\lambda}{4}$, where $g(a)=\frac{a}{a+\lambda}+\frac{(1-a)^2}{2(\lambda+1)(a+\lambda)}$. Combining over a common denominator, the numerator simplifies to

$$2(\lambda+1)a+(1-a)^2=a^2+2\lambda a+1=(a+\lambda)^2+(1-\lambda^2),$$

so

$$g(a)=\frac{a+\lambda}{2(\lambda+1)}+\frac{1-\lambda}{2(a+\lambda)}.$$

Summing and using $a+b+c=1$, we obtain

$$\sum_{cyc} g(a) = \frac{1+3\lambda}{2(\lambda+1)} + \frac{1-\lambda}{2}\sum_{cyc}\frac{1}{a+\lambda}.$$

Since $\lambda \geq 1$, we have $1-\lambda \leq 0$ while $\sum_{cyc}\frac{1}{a+\lambda} > 0$, so

$$\sum_{cyc} g(a) \leq \frac{1+3\lambda}{2(\lambda+1)}.$$

Moreover $\lambda \geq 1 \Rightarrow 2(\lambda+1) \geq 4$, and $1+3\lambda > 0$, so

$$\frac{1+3\lambda}{2(\lambda+1)} \leq \frac{1+3\lambda}{4}.$$

Combining the last two displays gives $\sum_{cyc} g(a) \leq \frac{1+3\lambda}{4}$, which proves (1) and hence the claim. Equality holds iff $\lambda = 1$ and $a = b = c = \frac{1}{3}$.