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SP.625 If $n \geq 3$ is an integer, then prove that $\sqrt{n+2} + \sqrt{n-2}$ is an irrational number.

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Solution by proposer

We argue by contradiction. Suppose that

$$\sqrt{n+2} + \sqrt{n-2} = \frac{p}{q}, \quad (p, q) = 1.$$

Inverting the relation gives

$$\frac{q}{p} = \frac{1}{\sqrt{n+2} + \sqrt{n-2}} = \frac{\sqrt{n+2} - \sqrt{n-2}}{4}$$

Hence we get the system of equations

$$\begin{aligned}\sqrt{n+2} + \sqrt{n-2} &= \frac{p}{q}, \\ \sqrt{n+2} - \sqrt{n-2} &= \frac{4q}{p}.\end{aligned}$$

Adding these equations and dividing by 2 gives $\sqrt{n+2} = \frac{4q^2 + p^2}{2pq}$.

Squaring and rearranging terms, we get $4np^2q^2 = 16q^4 + p^4$. Suppose now that n, p and q are all positive integers with p and q relatively prime. The relationship

$4np^2q^2 = 16q^4 + p^4$ shows that p^4 , and hence p , is divisible by 2. Letting $p = 2r$ we obtain $nr^2q^2 = q^4 + r^4$ from which we obtain $n = \left(\frac{q^2}{r^2}\right) + \left(\frac{r^2}{q^2}\right)$ and this occurs only when $\frac{q}{r} = \pm 1$ and $n = \pm 2$ and this contradicts that $n \geq 3$.

Alternatively, from $4np^2q^2 = 16q^4 + p^4$ we get $p^4 = q^2(4np^2 - 16q^2)$ and

$\left(\frac{p^2}{q}\right)^2 = 4np^2 - 16q^2$ and q divides p^2 which is not possible because $q > 1$. This

contradicts the assumption that p and q are coprime. Thus, the number $\sqrt{n+2} + \sqrt{n-2}$ is irrational.