ROMANIAN MATHEMATICAL MAGAZINE

SP.624 Let ABC be a triangle. Prove that:

$$\left(\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}\right)\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)\geq 3$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

In what follows, we find a lower bound for each term in the LHS.

1. The sum of the inverse squares of sines is bounded below by

$$\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}\geq 4$$

Indeed, using the fundamental Pythagorean trigonometric identity

$\csc^2\theta=1+\cot^2\theta$ (where $\csc\theta=\frac{1}{\sin\theta}$), we can rewrite each term in the sum as

$$\frac{1}{\sin^2 A}=1+\cot^2 A$$
$$\frac{1}{\sin^2 B}=1+\cot^2 B$$
$$\frac{1}{\sin^2 C}=1+\cot^2 C$$

Summing these three equations together gives

$$\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}=3+(\cot^2 A+\cot^2 B+\cot^2 C)$$

It is a well-known geometric and algebraic identity for the angles of any triangle that the sum of the squared cotangents satisfies

$$\cot^2 A+\cot^2 B+\cot^2 C\geq 1$$

Equality holds if and only if $\cot A=\cot B=\cot C=\frac{1}{\sqrt{3}}$, which corresponds to an equilateral triangle where $A=B=C=60°$.

Thus, we have

$$\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}\geq 3+1=4$$

2. The second factor of the LHS satisfies

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\geq\frac{3}{4}$$

Indeed,

1. **For any triangle ABC with interior angles A, B, C, the sum of half – angles satisfies the well-known trigonometric identity**

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

2. **It is also known the geometric property for triangle half-angles that the product of their sines is bounded above by $\frac{1}{8}$. That is,**

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$$

(Equality holds precisely when $\frac{A}{2}=\frac{B}{2}=\frac{C}{2}=30^\circ$, that is when ΔABC is equilateral triangle).

3. **Substituting this upper bound into our identity, since we are subtracting the product, the maximum product yields the minimum possible value for the sum**

$$1-2\left(\frac{1}{8}\right)=1-\frac{1}{4}=\frac{3}{4}$$

An alternative way to obtain the same bound is:

1. **Using the half-angle power – reducing identity $\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}$, we rewrite the sum for all three angles:**

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\frac{1-\cos A}{2}+\frac{1-\cos B}{2}+\frac{1-\cos C}{2}$$

2. **Combining the terms gives:**

$$=\frac{3-(\cos A+\cos B+\cos C)}{2}$$

3. **For any triangle, the sum of the cosines of its interior angles has a maximum value of $\frac{3}{2}$-That is,**

$$\cos A+\cos B+\cos C\leq\frac{3}{2}$$

4. To find the minimum value of the fraction, we substitute the maximum possible value for the subtracted term $\left(\frac{3}{2}\right)$ and we get

$$\frac{3-\frac{3}{2}}{2}=\frac{3-\frac{3}{2}}{2}=\frac{3}{4}$$

Multiplying our lower bounds together gives

$$\left(\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}\right)\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)\geq 4\times\frac{3}{4}=3$$

Equality case occurs when $A=B=C=60^\circ$. That is, when ΔABC is equilateral.

Solution 2 by Marin Chirciu-Romania

1. $\sum\frac{1}{\sin^2 A}=\frac{p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2}{4p^2r^2}\overset{Gerretsen}{\geq}4.$

2. $\sum\sin^2\frac{A}{2}=1-\frac{r}{2R}\overset{Euler}{\geq}\frac{3}{4}.$

$$LHS=\sum\frac{1}{\sin^2 A}\sum\sin^2\frac{A}{2}\geq 4\cdot\frac{3}{4}=3=RHS$$

Equality if and only if the triangle is equilateral.

Remark.

The problem can be developed and strengthened.

In ΔABC holds:

$$\frac{3R}{2r}\leq\sum\frac{1}{\sin^2 A}\sum\sin^2\frac{A}{2}\leq\frac{R}{r}\left(\frac{R}{r}-\frac{1}{2}\right)$$

Marin Chirciu

Solution

1. $\frac{2R}{r}\leq\sum\frac{1}{\sin^2 A}\leq\frac{R^2}{r^2}$, see $\sum\frac{1}{\sin^2 A}=\frac{p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2}{4p^2r^2}$ and the Gerretsen's inequality

$$16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2.$$

2. $\sum\sin^2\frac{A}{2}=1-\frac{r}{2R}\overset{Euler}{\geq}\frac{3}{4}.$

Right hand inequality.

$$\sum\frac{1}{\sin^2 A}\sum\sin^2\frac{A}{2}\leq\frac{R^2}{r^2}\left(1-\frac{r}{2R}\right)=\frac{R}{r}\left(\frac{R}{r}-\frac{1}{2}\right).$$

Equality holds if and only if the triangle is equilateral.

Right hand inequality.

$$\sum \frac{1}{\sin^2 A} \sum \sin^2 \frac{A}{2} \geq \frac{2R}{r} \cdot \frac{3}{4} = \frac{3R}{2r}.$$

Equality holds if and only if the triangle is equilateral.

Remark.

Left hand inequality strengthen Problem SP.624 from RMM Number 42 – Autumn 2026.

We can write the inequalities.

In ΔABC holds

$$3 \leq \frac{3R}{2r} \leq \sum \frac{1}{\sin^2 A} \sum \sin^2 \frac{A}{2} \leq \frac{R}{r}\left(\frac{R}{r} - \frac{1}{2}\right)$$