SP.623 Compute the following limit:

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\left(\frac{2k^2+3n^2}{3k^2+2n^2}\right)$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

To compute the limit, we can transform the sum into a Riemann sum and evaluate it using definite integral. Since

$$\frac{k}{n^2}\cdot\frac{2k^2+3n^2}{3k^2+2n^2}=\left(\frac{k}{n}\right)\cdot\frac{2\left(\frac{k}{n}\right)^2+3}{3\left(\frac{k}{n}\right)^2+2}\cdot\frac{1}{n}$$

then

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left[\left(\frac{k}{n}\right)\cdot\frac{2\left(\frac{k}{n}\right)^2+3}{3\left(\frac{k}{n}\right)^2+2}\cdot\frac{1}{n}\right]$$

This expression is precisely the Riemann sum for the function $f(x)=x\cdot\frac{2x^2+3}{3x^2+2}$ on the interval $[0,1]$ with step size $\Delta x=\frac{1}{n}$.

Taking the limit as $n\to+\infty$ converts the sum into the integral of $f(x)$ on $[0,1]$, and we have

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left[\left(\frac{k}{n}\right)\cdot\frac{2\left(\frac{k}{n}\right)^2+3}{3\left(\frac{k}{n}\right)^2+2}\cdot\frac{1}{n}\right]=\int_0^1 x\cdot\frac{2x^2+3}{3x^2+2}dx$$

Since

$$\int x\frac{2x^2+2}{3x^2+2}dx=\frac{1}{3}x^2+\frac{5}{18}\ln(3x^2+2)+C$$

then

$$\int_0^1 x\cdot\frac{2x^2+3}{3x^2+2}dx=\left[\frac{1}{3}x^2+\frac{5}{18}\ln(3x^2+2)+C\right]_0^1=\frac{1}{3}+\frac{5}{18}\ln\left(\frac{5}{2}\right)$$

and

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\left(\frac{2k^2+3n^2}{3k^2+2n^2}\right)=\frac{1}{3}+\frac{5}{18}\ln\left(\frac{5}{2}\right)$$

Solution 2 by Marin Chirciu-Romania

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\left(\frac{2k^2+3n^2}{3k^2+2n^2}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\left(\frac{2\frac{k^2}{n^2}+3}{3\frac{k^2}{n^2}+2}\right)=\int_0^1 x\frac{2x^2+3}{3x^2+2}dx=$$

$$=\int_0^1\frac{2x^3+3x}{3x^2+2}dx=$$

$$=\frac{1}{3}\int_0^1\frac{6x^3+9x}{3x^2+2}dx=\frac{1}{3}\int_0^1\left(2x+\frac{5x}{3x^2+2}\right)dx=\frac{1}{3}\left(\int_0^1 2x\,dx+\int_0^1\frac{5x}{3x^2+2}dx\right)=$$

$$=\frac{1}{3}\left(\int_0^1 2x\,dx+\frac{5}{6}\int_0^1\frac{6x}{3x^2+2}dx\right)=\frac{1}{3}\left(x^2|_0^1+\frac{5}{6}\ln(3x^2+2)\Big|_0^1\right)=\frac{1}{3}\left(1+\frac{5}{6}\ln\frac{5}{2}\right).$$

We deduce that $\lim\limits_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\left(\frac{2k^2+3n^2}{3k^2+2n^2}\right)=\frac{1}{3}\left(1+\frac{5}{6}\ln\frac{5}{2}\right)$.

Remark: The problem can be developed.

Compute the following limit:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\left(\frac{\lambda k^2+(\lambda+1)n^2}{(\lambda+1)k^2+\lambda n^2}\right),\lambda>0.$$

Marin Chirciu

Solution:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\left(\frac{\lambda k^2+(\lambda+1)n^2}{(\lambda+1)k^2+\lambda n^2}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\left(\frac{\lambda\frac{k^2}{n^2}+\lambda+1}{(\lambda+1)\frac{k^2}{n^2}+\lambda}\right)=$$

$$=\int_0^1 x\frac{\lambda x^2+\lambda+1}{(\lambda+1)x^2+\lambda}dx=$$

$$=\int_0^1\frac{\lambda x^3+(\lambda+1)x}{(\lambda+1)x^2+\lambda}dx=\frac{1}{\lambda+1}\int_0^1\frac{\lambda(\lambda+1)x^3+(\lambda+1)^2x}{(\lambda+1)x^2+\lambda}dx=$$

$$=\frac{1}{\lambda+1}\int_0^1\left(\lambda x+\frac{(2\lambda+1)x}{(\lambda+1)x^2+\lambda}\right)dx=\frac{1}{\lambda+1}\left(\int_0^1\lambda x\,dx+\int_0^1\frac{(2\lambda+1)x}{(\lambda+1)x^2+\lambda}dx\right)=$$

$$=\frac{1}{\lambda+1}\left(\int_0^1\lambda x\,dx+\frac{2\lambda+1}{2(\lambda+1)}\int_0^1\frac{2(\lambda+1)x}{(\lambda+1)x^2+\lambda}dx\right)=$$

$$=\frac{1}{\lambda+1}\left(\frac{\lambda x^2}{2}\Big|_0^1+\frac{2\lambda+1}{2(\lambda+1)}\ln\left((\lambda+1)x^2+\lambda\right)\Big|_0^1\right)=\frac{1}{\lambda+1}\left(\frac{\lambda}{2}+\frac{2\lambda+1}{2(\lambda+1)}\ln\frac{2\lambda+1}{\lambda}\right).$$

We deduce that $\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\left(\frac{\lambda k^2+(\lambda+1)n^2}{(\lambda+1)k^2+\lambda n^2}\right)=\frac{1}{\lambda+1}\left(\frac{\lambda}{2}+\frac{2\lambda+1}{2(\lambda+1)}\ln\frac{2\lambda+1}{\lambda}\right).$

Note: For $\lambda=2$ we obtain Problem SP.623 from RMM Number 42 – Autumn 2026.

SP.623. Compute the following limit

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\left(\frac{2k^2+3n^2}{3k^2+2n^2}\right).$$

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