ROMANIAN MATHEMATICAL MAGAZINE

SP.622 Compute the value of the integral:

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{\pi^{\frac{1}{t}} + 1} dt$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

To compute the value of the integral, we can use the substitution property for definite integrals (King's Property):

$$\int_a^b f(t)\,dt = \int_a^b f(a+b-t)\,dt$$

For our integral, $a = -\frac{\pi}{4}$ and $b = \frac{\pi}{4}$, so $a + b = 0$. Substituting t with $-t$ gives

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos(-t)}{\pi^{\frac{1}{-t}} + 1}(-dt) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{\pi^{-\frac{1}{t}} + 1} dt$$

Since $\pi^{-\frac{1}{t}} = \frac{1}{\pi^{\frac{1}{t}}}$, the denominator simplifies as follows

$$\frac{1}{\pi^{-\frac{1}{t}} + 1} = \frac{\pi^{\frac{1}{t}}}{\pi^{\frac{1}{t}} + 1}$$

Thus, the integral can be rewritten as

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t \cdot \pi^{\frac{1}{t}}}{\pi^{\frac{1}{t}} + 1} dt$$

Adding our original expression for I and this new expression together yields

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{\pi^{\frac{1}{t}} + 1} dt + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t \cdot \pi^{\frac{1}{t}}}{\pi^{\frac{1}{t}} + 1} dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos t\, dt$$

Now, we evaluate this straightforward integral to obtain

$$2I = [\sin t]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \sqrt{2} \Rightarrow I = \frac{\sqrt{2}}{2}$$

Solution 2 by Angel Plaza-Spain

Let $f(t)=\frac{\cos t}{\pi^{\frac{1}{t}}+1}$. Then

$$f(t)+f(-t)=\frac{\cos t}{\pi^{\frac{1}{t}}+1}+\frac{\cos(-t)}{\pi^{\frac{-1}{t}}+1}=\cos t\left(\frac{1}{\pi^{\frac{1}{t}}+1}+\frac{\pi^{\frac{1}{t}}}{1+\pi^{\frac{1}{t}}}\right)=\cos t$$

$$I=\int_0^{\frac{\pi}{4}}\cos t\,dt=\sin t\Big]_0^{\frac{\pi}{4}}=\frac{\sqrt{2}}{2}.$$

Solution 3 by Marin Chirciu – Romania

$$I=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\pi^{\frac{1}{t}}}dt\overset{t\to-1}{=}\int_{\frac{\pi}{4}}^{\frac{-\pi}{4}}\frac{\cos(-t)}{1+\pi^{\frac{1}{-t}}}(-dt)=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\pi^{\frac{-1}{t}}}dt=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\frac{1}{\pi^{\frac{1}{t}}}}dt=$$

$$=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\pi^{\frac{1}{t}}\cos t}{1+\pi^{\frac{1}{t}}}dt.$$

We obtain:

$$I+I=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\pi^{\frac{1}{t}}}dt+\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\pi^{\frac{1}{t}}\cos t}{1+\pi^{\frac{1}{t}}}dt=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\left(1+\pi^{\frac{1}{t}}\right)\cos t}{1+\pi^{\frac{1}{t}}}dt=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\cos t\,dt=\sin t\Big|_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$=$$

$$=\sin\frac{\pi}{4}-\sin\frac{-\pi}{4}=2\sin\frac{\pi}{4}=2\cdot\frac{\sqrt{2}}{2}=\sqrt{2}$$

From $2I=\sqrt{2}\Rightarrow I=\frac{\sqrt{2}}{2}$.

We deduce that $I=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\pi^{\frac{1}{t}}}dt=\frac{\sqrt{2}}{2}$.

Remark: The problem can be developed.

Let be $\lambda>0$ fixed. Compute the value of the integral

$$I=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\lambda^{\frac{1}{t}}}dt.$$

Marin Chirciu

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Solution

$$I=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\lambda^{\frac{1}{t}}}dt\overset{t\to -t}{=}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos(-t)}{1+\lambda^{\frac{1}{-t}}}(-dt)=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\lambda^{\frac{-1}{t}}}dt=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\frac{1}{\lambda^{\frac{1}{t}}}}dt=$$

$$=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\lambda^{\frac{1}{t}}\cos t}{1+\lambda^{\frac{1}{t}}}dt.$$

We obtain :

$$I+I=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\lambda^{\frac{1}{t}}}dt+\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\lambda^{\frac{1}{t}}\cos t}{1+\lambda^{\frac{1}{t}}}dt+\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\left(1+\lambda^{\frac{1}{t}}\right)\cos t}{1+\lambda^{\frac{1}{t}}}dt=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\cos t\,dt=\sin t\Big|_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$=$$

$$=\sin\frac{\pi}{4}-\sin\frac{-\pi}{4}=2\sin\frac{\pi}{4}=2\cdot\frac{\sqrt{2}}{2}=\sqrt{2}.$$

From $2I=\sqrt{2}\Rightarrow I=\frac{\sqrt{2}}{2}$.

We deduce that $I=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\lambda^{\frac{1}{t}}}dt=\frac{\sqrt{2}}{2}$.

Note: For $\lambda=\pi$ we obtain Problem SP.622. from RMM Number 42 – Autumn 2026.

SP.622. Compute the value of the integral

$$I=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\cos t}{1+\pi^{\frac{1}{t}}}dt$$

Jose Luis Diaz – Barrero – Spain.