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SP.621 Prove that $\frac{5}{4}$ is the least positive value of the power k such that the inequality $(ab)^k + (ac)^k + (ad)^k + (bc)^k + (bd)^k + (cd)^k \geq 6$ holds for any nonnegative real numbers a, b, c, d with $a \geq b \geq c \geq 1 \geq d$ and $ab + bc + cd + da = 4$.

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Solution by proposer

For $a = b$ and $c = 1$, we have $(a + 1)(a + d) = 4$, and the inequality becomes $E \geq 0$, where

$$E = a^{2k} + 2a^k(1 + d^k) + d^k - 6$$

Assume that a and E are functions of d . If $d = 1$, then $a = 1$. We have

$$(2a + d + 1)a' + a + 1 = 0, \quad a'(1) = -\frac{1}{2},$$

$$(2a + d + 1)a'' + 2(a')^2 + 2a' = 0, \quad a''(1) = \frac{1}{8},$$

$$\frac{E'(d)}{k} = 2[a^{2k-1} + a^{k-1}(1 + d^k)]a' + 2a^k d^{k-1} + d^{k-1}, \quad E'(1) = 0,$$

$$\begin{aligned} \frac{E''(d)}{k} = & 2[a^{2k-1} + a^{k-1}(1 + d^k)]a'' + 2[(2k-1)a^{2k-2} + (k-1)a^{k-2}(1 + d^k)](a')^2 \\ & + 4ka^{k-1}a' + 2(k-1)a^k d^{k-2} + (k-1)d^{k-2}, \end{aligned}$$

hence

$$\frac{E''(1)}{k} = \frac{3}{4} + \frac{8k-6}{4} - 2k + 2(k-1) + k - 1 = \frac{3(4k-5)}{4}.$$

Since $E(1) = E'(1) = 0$, the condition $E''(1) \geq 0$ is necessary to have $E(d) \geq 0$ for

$d \in [0, 1]$. This condition implies $k \geq \frac{5}{4}$. So, it suffices to prove the required inequality for

$k = \frac{5}{4}$. Replacing a, b, c, d with a^4, b^4, c^4, d^4 , respectively, we need to show that $F \geq 0$ for

$$(a^4 + c^4)(b^4 + d^4) = 4, \quad a \geq b \geq c \geq 1 \geq d \geq 0,$$

where

$$F(ab)^5 + (ac)^5 + (ad)^5 + (bc)^5 + (bd)^5 + (cd)^5 - 6.$$

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For fixed a and b , assume that d and F are functions of c . Since

$$d^3(a^4 + c^4)d' + c^3(b^4 + d^4) = 0,$$

we have

$$\begin{aligned} \frac{F'(c)}{5} &= c^4(a^5 + b^5 + d^5) + d^4(a^5 + b^5 + c^5)d' = c^4(a^5 + b^5 + d^5) - \\ &\quad - \frac{c^3 d(a^5 + b^5 + c^5)(b^4 + d^4)}{a^4 + c^4}. \end{aligned}$$

We will show that $F'(c) \geq 0$, which is equivalent to $G \geq 0$, where

$$G = c(a^5 + b^5 + d^5)(a^4 + c^4) - d(a^5 + b^5 + c^5)(b^4 + d^4).$$

For fixed b, c and d , we have

$$G(a) = a^5[c(a^4 + c^4) - d(b^4 + d^4)] + (b^5 + d^5)c(a^4 + c^4) - (b^5 + c^5)d(b^4 + d^4).$$

Since $c(a^4 + c^4) - d(b^4 + d^4) \geq 0$, $G(a)$ is increasing and

$$\begin{aligned} G(a) &\geq G(b) = b^5[c(b^4 + c^4) - d(b^4 + d^4)] + \\ &\quad + (b^5 + d^5)c(b^4 + c^4) - (b^5 + c^5)d(b^4 + d^4) \\ &= 2(2b^5 + d^5)c(b^4 + c^4) - (2b^5 + c^5)d(b^4 + d^4) = b^4 A, \end{aligned}$$

where

$$A = 2b^5(c - d) + 2b(c^5 - d^5) - cd(c^4 - d^4).$$

Since

$$A \geq 2b(c^5 - d^5) - cd(c^4 - d^4) \geq c(c^5 - d^5) - cd(c^4 - d^4) = c^5(c - d) \geq 0,$$

we have $G(a) \geq 0$, hence $F'(c) \geq 0$, $F(c)$ is increasing and has the minimum value when c is minimum, hence when $c = 1$. So, it suffices to consider the case $c = 1$. We need to show that $H \geq 0$ for $a \geq b \geq 1 \geq d$ such that $(a^4 + 1)(b^4 + d^4) = 4$, where

$$H = (ab)^5 + (bd)^5 + (da)^5 + a^5 + b^5 + d^5 - 6.$$

From $4 = (a^4 + 1)(b^4 + d^4) > a^4 b^4 \geq b^8$, we $b < 2$. For fixed d , assume that b and H are functions of a . Since

$$b^3(a^4 + 1)b' + a^3(b^4 + d^4) = 0,$$

we have

$$\frac{H'(a)}{5} = a^4(b^5 + d^5 + 1) + b^4(a^5 + d^5 + 1)b' = a^4(b^5 + d^5 + 1) -$$

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$$- \frac{a^3 b(a^5 + d^5 + 1)(b^4 + d^4)}{a^4 + 1}.$$

We will show that $H'(a) \geq 0$, which is equivalent to $K \geq 0$, where

$$\begin{aligned} K &= a(a^4 + 1)(b^5 + d^5 + 1) - b(b^4 + d^4)(a^5 + d^5 + 1) \\ &= (d^5 + 1 - bd^4)a^5 + (b^5 + d^5 + 1)a - b(b^4 + d^4)(d^5 + 1). \end{aligned}$$

Since

$$d^5 + 1 - bd^4 \geq d^8 + 1 - bd^4 \geq 2d^4 - bd^4 = d^4(2 - b) \geq 0,$$

we have

$$\begin{aligned} K &\geq (d^5 + 1 - bd^4)b^5 + (b^5 + d^5 + 1)b - b(b^4 + d^4)(d^5 + 1) = \\ &= b(b^5 + d^5 + 1)(1 - d^4) \geq 0. \end{aligned}$$

So, $H'(a) \geq 0$, $H(a)$ is increasing and has the minimum value when a is minimum, hence when $a = b$. Thus, it suffices to consider the case $a = b$, when we need to show that $(b^4 + 1)(b^4 + d^4) = 4$ implies

$$b^{10} + 2(d^5 + 1)b^5 + d^5 \geq 6.$$

Write this inequality as

$$d^5 \geq \frac{6 - 2b^5 - b^{10}}{1 + 2b^5}.$$

Consider the nontrivial case when $6 - 2b^5 - b^{10} \geq 0$, hence when $1 \leq b^5 \leq \sqrt{7} - 1 < \sqrt{3}$. The inequality is true if

$$(d^4)^5 \geq \left(\frac{6 - 2b^5 - b^{10}}{1 + 2b^5} \right)^4,$$

that is

$$\left(\frac{4 - b^4 - b^8}{1 + b^4} \right)^5 \geq \left(\frac{6 - 2b^5 - b^{10}}{1 + 2b^5} \right)^4.$$

We will show that

$$\left(\frac{4 - b^4 - b^8}{1 + b^4} \right)^5 \geq \left(\frac{101 + 18b^5 - 44b^{10}}{25(1 + 2b^5)} \right)^5 \geq \left(\frac{6 - 2b^5 - b^{10}}{1 + 2b^5} \right)^4.$$

The left inequality is true if

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$$\frac{4 - b^4 - b^8}{1 + b^4} \geq \frac{101 + 18b^5 - 44b^{10}}{25(1 + 2b^5)},$$

which is equivalent to

$$\begin{aligned} &44b^{14} - 50b^{13} + 44b^{10} - 68b^9 - 25b^8 + 182b^5 - 126b^4 - 1 \geq 0, \\ &(b - 1)^3 \left(44b^{11} + 82b^{10} + 114b^9 + 140b^8 + 204b^7 + 238b^6 + 217b^5 + 141b^4 + \right. \\ &\quad \left. + 10b^3 + 6b^2 + 3b + 1 \right) \\ &\geq 0. \end{aligned}$$

Denoting $t = b^5 \in [1, \sqrt{3})$, the right inequality becomes

$$\left(\frac{101 + 18t - 44t^2}{25} \right)^5 \geq (1 + 2t)(6 - 2t - t^2)^4.$$

Since $101 + 18t - 44t^2 > 0$, it is true if $g(t) \geq 0$, where

$$g(t) = 5 \ln(101 + 18t - 44t^2) - 5 \ln 25 - \ln(1 + 2t) - 4 \ln(6 - 2t - t^2).$$

From

$$\begin{aligned} \frac{g'(t)}{2} &= \frac{5(9 - 44t)}{101 + 18t - 44t^2} - \frac{1}{1 + 2t} + \frac{4(1 + t)}{6 - 2t - t^2} \\ &= \frac{4(11t^4 + 139t^3 - 294t^2 + 127t + 17)}{(101 + 18t - 44t^2)(1 + 2t)(6 - 2t - t^2)} \\ &= \frac{4(t - 1)^2(11t^2 + 161t + 17)}{(101 + 18t - 44t^2)(1 + 2t)(6 - 2t - t^2)} \geq 0, \end{aligned}$$

it follows that $g(t)$ is increasing, hence $g(t) \geq g(1) = 0$.

For $k = \frac{5}{4}$, the equality occurs when $a = b = c = d = 1$.