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SP.620 Let a, b, c be nonnegative real numbers such that at most one of them is larger than 1 and $ab + bc + ca = 3$. Prove that:

$$8 \left(\sqrt{1 + 3a^2} + \sqrt{1 + 3b^2} + \sqrt{1 + 3c^2} \right) \leq 21 + 3(a + b + c)^2.$$

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Solution by proposer

Assume that $a \geq 1 \geq b \geq c$ and write the inequality as $E \leq 0$, where

$$E = 8 \left(\sqrt{1 + 3a^2} + \sqrt{1 + 3b^2} + \sqrt{1 + 3c^2} \right) - 21 - 3(a + b + c)^2.$$

If $a = c$, then $a = b = c = 1$ and $E = 0$. Assume further $a > c$ and denote $p = a + b + c$. For fixed p , consider that a, c and E are functions of b . By differentiating the equality constraints $a + b + c = p$ and $ab + bc + ca = 3$, we get

$$a' + c' + 1 = 0, \quad (p - 1)a' + (p - c)c' + p - b = 0,$$

$$\text{hence } a'(b) = \frac{-(b-c)}{a-c}, \quad c'(b) = \frac{-(a-b)}{a-c}.$$

Since $a'(b) \leq 0$ and $c'(b) \leq 0$, the functions $a(b)$ and $c(b)$ are decreasing. We have

$$\frac{E'(b)}{24} = \frac{aa'}{\sqrt{1 + 3a^2}} + \frac{cc'}{\sqrt{1 + 3c^2}} + \frac{b}{\sqrt{1 + 3b^2}},$$

therefore

$$\begin{aligned} \frac{-(a-c)E'(b)}{24} &= \frac{(b-c)a}{\sqrt{1 + 3a^2}} + \frac{(a-b)c}{\sqrt{1 + 3c^2}} - \frac{(a-c)b}{\sqrt{1 + 3b^2}} = \\ &= (b-c)f(a) + (a-b)f(c) - (a-c)f(b), \end{aligned}$$

where $f(x) = \frac{x}{\sqrt{1+3x^2}}$. Since $f''(x) = \frac{-9x}{(1+3x^2)^{\frac{5}{2}}} \leq 0$,

$f(x)$ is concave on $[0, \infty)$. By Jensen's inequality, we have

$$\begin{aligned} (b-c)f(a) + (a-b)f(c) &\leq [(b-c) + (a-b)]f\left(\frac{(b-c)a + (a-b)c}{(b-c) + (a-b)}\right) \\ &= (a-c)f(b), \end{aligned}$$

hence $E'(b) \geq 0$. So, $E(b)$ is increasing and has the maximum value when b is maximum (or a or c are minimum), hence when $b = 1$ or $c = 0$. Thus, it suffices to consider these cases.

Case 1: $b = 1$. We need to show that $a + c + ac = 3$ implies

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$$8\left(\sqrt{1+3a^2} + \sqrt{1+3c^2}\right) \leq 5 + 3(1+a+c)^2.$$

Let $x = \frac{a+c}{2}$. From $3 = 2x + ac \leq 2x + x^2$, we get $x \geq 1$. By squaring, the required inequality becomes

$$8\left(6x^2 - 3ac + 1 + \sqrt{9a^2c^2 - 6ac + 12x^2 + 1}\right) \leq (3x^2 + 3x + 2)^2,$$

$$32\sqrt{3x^2 - 6x + 4} \leq 9x^4 + 18x^3 - 27x^2 - 36x + 68.$$

Since

$$\begin{aligned} 9x^4 + 18x^3 - 27x^2 - 36x + 68 &= 9(x-1)^2(x^2 + 4x - 4) + 8(9x^2 - 18x + 13) \geq \\ &\geq 8(9x^2 - 18x + 13), \end{aligned}$$

it suffices to show that

$$4\sqrt{3x^2 - 6x + 4} \leq 9x^2 - 18x + 13.$$

Squaring again, the inequality becomes

$$\begin{aligned} (9x^2 - 18x + 13)^2 &\geq 16(3x^2 - 6x + 4), \\ 27x^4 - 108x^3 + 170x^2 - 124x + 35 &\geq 0, \\ (x-1)^2(27x^2 - 54x + 35) &\geq 0. \end{aligned}$$

It is true because $27x^2 - 54x + 35 > 27x^2 - 54x + 27 = 27(x-1)^2 \geq 0$.

Case 2: $c = 0$. We need to show that $ab = 3$ implies

$$8\left(\sqrt{1+3a^2} + \sqrt{1+3b^2}\right) \leq 20 + 3(a+b)^2.$$

Replacing a and b with $a\sqrt{3}$ and $b\sqrt{3}$, respectively, we need to show that $ab = 1$ implies

$$8\left(\sqrt{1+9a^2} + \sqrt{1+9b^2}\right) \leq 20 + 9(a+b)^2.$$

Since

$$\sqrt{1+9a^2} + \sqrt{1+9b^2} \leq \sqrt{2(1+9a^2+1+9b^2)},$$

it suffices to show that

$$8\sqrt{2+18a^2+18b^2} \leq 20+9(a+b)^2$$

Denoting $x = \frac{a+b}{2}$, the inequality becomes

$$\begin{aligned} 2\sqrt{72x^2 - 34} &\leq 9x^2 + 5, \quad 4(72x^2 - 34) \leq (9x^2 + 5)^2, \\ 81x^4 - 198x^2 + 161 &\geq 0. \end{aligned}$$

We have: $81x^4 - 198x^2 + 161 > 81x^4 - 198x^2 + 121 = (9x^2 - 11)^2 \geq 0$.

The proof is finished. The equality occurs for $a = b = c = 1$.