

# ROMANIAN MATHEMATICAL MAGAZINE

SP.619 Let  $a, b, c, d, e$  be positive real numbers such that  $a \geq b \geq c \geq d \geq e$  and  $abcde = 1$ . Prove that:

$$\frac{1}{3ab+2} + \frac{1}{3bc+2} + \frac{1}{3cd+2} + \frac{1}{3de+2} + \frac{1}{3ea+2} \geq 1$$

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*Solution by proposer*

Write the inequality as  $E \geq 0$ , where

$$E = \frac{1}{3ab+2} + \frac{1}{3bc+2} + \frac{1}{3cd+2} + \frac{1}{3de+2} + \frac{1}{3ea+2} - 1$$

For fixed  $a, d$  and  $e$ , we may consider that  $b$  and  $E$  are functions of  $c$ . Since

$$bc = \frac{1}{ade} = \text{fixed}, \text{ we have } cb' + b = 0 \text{ and}$$

$$\begin{aligned} \frac{E'(c)}{3} &= \frac{-ab'}{(3ab+2)^2} - \frac{d}{(3cd+2)^2} = \frac{ab}{c(3ab+2)^2} - \frac{d}{(3cd+2)^2} = \\ &= \frac{(ab-cd)(4-9abcd)}{c(3ab+2)^2(3cd+2)^2} \leq 0 \end{aligned}$$

So,  $E(c)$  is decreasing and has the minimum value when  $c$  is maximum, hence when  $c = b$ . Therefore, it suffices to prove the required inequality for  $c = b$ , when we need to

show that

$$\frac{1}{3ab+2} + \frac{1}{3b^2+2} + \frac{1}{3bd+2} + \frac{1}{3de+2} + \frac{1}{3ae+2} \geq 1$$

for  $a \geq b \geq d \geq e$  such that  $ab^2de = 1$ . Denote  $x = de$  and  $y = bd$ . Since  $ab = \frac{1}{bx}$  and

$ae = \frac{1}{by}$  the inequality becomes

$$\frac{bx}{2bx+3} + \frac{1}{3b^2+2} + \frac{1}{3y+2} + \frac{1}{3x+2} + \frac{by}{2by+3} \geq 1,$$

that is

$$f(x) + f(y) + \frac{1}{3b^2+2} \geq 1,$$

$$\text{where } f(t) = \frac{bt}{2bt+3} + \frac{1}{3t+2}$$

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**Case 1:**  $b \geq \frac{9}{4}$ . By Lemma below, we have  $f(x) \geq \frac{1}{2}$  and  $f(y) \geq \frac{1}{2}$ . So, the required inequality is true.

**Case 2:**  $1 \leq b \leq \frac{9}{4}$ . By Lemma, we have

$$f(x) \geq \frac{2\sqrt{b}}{2\sqrt{b}+3}, \quad f(y) \geq \frac{2\sqrt{b}}{2\sqrt{b}+3}.$$

So, it suffices to show that

$$\frac{4\sqrt{b}}{2\sqrt{b}+3} + \frac{1}{3b^2+2} \geq 1.$$

Denoting  $z = \sqrt{b} \geq 1$ , the inequality becomes

$$\frac{4z}{2z+3} \geq \frac{3z^4+1}{3z^4+2}, \quad 2z^5 - 3z^4 + 2z - 1 \geq 0, \quad (z-1)^2(2z^3 + z^2 - 1) \geq 0.$$

**Case 3:**  $0 < b \leq 1$ . Since  $x = de \leq b^2$  and  $y = bd \leq b^2$ , by Lemma below we have

$$f(x) \geq \frac{b^3}{2b^3+3} + \frac{1}{3b^2+2}, \quad f(y) = \frac{b^3}{2b^3+3} + \frac{1}{3b^2+2}.$$

So, it suffices to show that

$$\frac{2b^3}{2b^3+3} + \frac{3}{3b^2+2} \geq 1$$

which is equivalent to

$$\frac{2b^3}{2b^3+3} \geq \frac{3b^2-1}{3b^2+2}, \quad 2b^3 - 3b^2 + 1 \geq 0, \quad (b-1)^2(2b+1) \geq 0.$$

Thus, the proof is completed. The equality occurs for  $a = b = c = d = e = 1$ .

**Lemma.** Let

$$f(t) = \frac{bt}{2bt+3} + \frac{1}{3t+2}, \quad t \geq 0.$$

(a) For  $b \geq \frac{9}{4}$ , we have  $f(t) \geq \frac{1}{2}$ .

(b) For  $1 \leq b \leq \frac{9}{4}$ , we have  $f(t) \geq f\left(\frac{1}{\sqrt{b}}\right) = \frac{2\sqrt{b}}{2\sqrt{b}+3}$ .

(c) For  $0 < b \leq 1$  and  $0 \leq t \leq b^2$ , we have  $f(t) \geq f(b^2) = \frac{b^3}{2b^3+3} + \frac{1}{3b^2+2}$ .

**Proof.**

We have

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$$\begin{aligned}\frac{f'(t)}{3} &= \frac{b}{(2bt+3)^2} - \frac{1}{(3t+2)^2} = \frac{[\sqrt{b}(3t+2) - 2bt - 3][\sqrt{b}(3t+2) + 2bt + 3]}{(2bt+3)^2(3t+2)^2} \\ &= \frac{(3 - 2\sqrt{b})(\sqrt{b} \cdot t - 1)[\sqrt{b}(3t+2) + 2bt + 3]}{(2bt+3)^2(3t+2)^2}.\end{aligned}$$

(a) We have

$$f(t) - \frac{1}{2} = \frac{bt}{2bt+3} - \frac{3t}{2(3t+2)} = \frac{t(4b-9)}{2(bt+3)(3t+2)} \geq 0.$$

(b) Since  $f'(t) \leq 0$  for  $0 \leq t \leq \frac{1}{\sqrt{b}}$  and  $f'(t) \geq 0$  for  $t \geq \frac{1}{\sqrt{b}}$ ,  $f(t)$  is decreasing on

$$\left[0, \frac{1}{\sqrt{b}}\right] \text{ and increasing on } \left[\frac{1}{\sqrt{b}}, \infty\right), \text{ hence } f(t) \geq f\left(\frac{1}{\sqrt{b}}\right).$$

(c) Since  $t \leq b^2 \leq \frac{1}{\sqrt{b}}$ , we have  $f'(t) \leq 0$ ,  $f(t)$  is decreasing on  $[0, b^2]$ , hence

$$f(t) \geq f(b^2).$$

Remark. Note that  $\frac{2}{3}$  is the largest positive value of  $k$  such that

$$\frac{1}{ab+k} + \frac{1}{bc+k} + \frac{1}{cd+k} + \frac{1}{de+k} + \frac{1}{ea+k} \geq \frac{5}{1+k}$$

for any positive numbers  $a, b, c, d, e$  with  $a \geq b \geq c \geq d \geq e$  and  $abcde = 1$ . For

$a = b = c = d := x$  and  $e = \frac{1}{x^4}$ , where  $x \geq 1$ , the constraints are satisfied, while the

inequality becomes

$$\frac{3}{x^2+k} + \frac{2x^3}{kx^3+1} \geq \frac{5}{1+k}.$$

For  $x \rightarrow \infty$ , we get the necessary condition  $k \leq \frac{2}{3}$ .