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SP.618 Prove that $\frac{17}{3}$ is the smallest positive value of the constant k such that

$$\frac{1}{a+k} + \frac{1}{b+k} + \frac{1}{c+k} + \frac{1}{d+k} \leq \frac{4}{1+k}$$

for any $a, b, c, d \geq \frac{1}{16}$ with $ab + ac + ad + bc + bd + cd = 6$.

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Solution by proposer

For $a = b = c = \frac{4}{3}$ and $d = \frac{1}{6}$, the constraints are satisfied and the inequality implies $k \geq \frac{17}{3}$. To show that $\frac{17}{3}$ is the smallest positive value of k , we need to prove that $E \leq 0$, where

$$E = \frac{1}{3a+17} + \frac{1}{3b+17} + \frac{1}{3c+17} + \frac{1}{3d+17} - \frac{1}{5}.$$

For fixed c and d , the expression E has the maximum value when

$$F(a, b) = \frac{1}{3a+17} + \frac{1}{3b+17}$$

is maximum. Assuming $a \geq b$ and denoting

$$a_1 = c + d, \quad b_1 = 6 - cd,$$

we have $a_1 > 0, b_1 > 0$ and $ab + a_1(a + b) = b_1$. By Lemma below, the expression

$F(a, b)$ has the maximum value when $b = \frac{1}{6}$ or $a = b$. So, it suffices to prove that $E \leq 0$

for these two cases. Similarly, applying Lemma for any two of the numbers a, b, c, d with $a, b, c, d \geq \frac{1}{6}$, it suffices to prove that $E \leq 0$ for

$$(a, b, c, d) = (1, 1, 1, 1), (a, b, c, d) = \left(x, x, x, \frac{1}{6}\right),$$

$(a, b, c, d) = \left(y, y, \frac{1}{6}, \frac{1}{6}\right)$, and $(a, b, c, d) = \left(z, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$, where $x = \frac{4}{3}, y = \frac{\sqrt{219}-2}{6} \approx 2.1331$

(with $36y^2 + 24y - 215 = 0$), and $z = \frac{71}{6}$. In the first two cases, we have $E = 0$. In the third case, we have

$$E(y) = \frac{2}{3y+17} - \frac{3}{35}.$$

Since $y > \frac{17}{8}$, we get

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$$E(y) < E\left(\frac{17}{8}\right) = \frac{16}{187} - \frac{3}{35} = \frac{-1}{6545} < 0.$$

In the last case, we have

$$E(z) = \frac{1}{3z+17} - \frac{1}{35} = \frac{2}{105} - \frac{1}{35} = \frac{-1}{105} < 0.$$

The proof is finished. If $a \geq b \geq c \geq d \geq \frac{1}{6}$, then the equality occurs for

$$(a, b, c, d) = (1, 1, 1, 1) \text{ and } (a, b, c, d) = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{1}{6}\right).$$

Lemma. Let a_1, b_1 and m be positive real constants, and let $a \geq b \geq m$ such that

$$ab + a_1(a + b) = b_1.$$

For $k > 0$, the expression

$$E = \frac{1}{a+k} + \frac{1}{b+k}$$

has extreme values for $a = b$ or $b = m$.

Proof. Denoting $s = \frac{a+b}{2}$, the equality constraint becomes $ab + 2a_1s = b_1$. From

$$b_1 = ab + 2a_1s \leq s^2 + 2a_1s,$$

it follows that s is minimum when $ab = s^2$, hence when $a = b$. From $0 \leq (a - m)(b - m)$ (with equality for $b = m$), we get $0 \leq ab - 2ms + m^2 - b_1 - 2a_1s - 2ms + m^2$, hence

$s \leq \frac{b_1+m^2}{2(a_1+m)}$; so, s is maximum when $b = m$. As a consequence, it suffices to show that the expression

$$E = \frac{2(s+k)}{b_1 + 2(k-a_1)s + k^2}$$

has extreme values when s is minimum or maximum. From

$$E'(s) = \frac{2(2ka_1 + b_1 - k^2)}{[b_1 + 2(k-a_1)s + k^2]^2},$$

it follows that $E(s)$ is increasing for $2ka_1 + b_1 - k^2 \geq 0$, and decreasing for

$2ka_1 + b_1 - k^2 \leq 0$. In both cases, $E(s)$ has extreme values when s is extreme, hence when $a = b$ or $b = m$.