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SP.617 Let a, b, c, d be positive real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $ab + ac + ad + bc + bd + cd = 6$. Prove that:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 \geq 3(a + b + c + d)$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Write the inequality as $E \geq 0$, where

$$E = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 - 3(a + b + c + d).$$

For fixed a and b , we may consider that c and E are functions of d . By differentiating the equality constraint, we get $(a + b + d)c' + a + b + c = 0$. So,

$$E'(d) = \left(-\frac{1}{c^2} - 3\right)c' - \frac{1}{d^2} - 3 = \left(\frac{1}{c^2} + 3\right)\frac{a + b + c}{a + b + d} - \frac{1}{d^2} - 3.$$

We will show that $E'(d) \leq 0$, that is

$$\frac{a + b + c}{a + b + d} \leq \frac{\frac{1}{d^2} + 3}{\frac{1}{c^2} + 3}, \frac{a + b + c}{a + b + d} - 1 \leq \frac{\frac{1}{d^2} + 3}{\frac{1}{c^2} + 3} - 1, \frac{c - d}{a + b + d} \leq \frac{(c - d)(c + d)}{(3c^2 + 1)d^2}.$$

We need to show that

$$\frac{1}{a + b + d} \leq \frac{c + d}{(3c^2 + 1)d^2}.$$

It is true because

$$\frac{1}{a + b + d} \leq \frac{1}{1 + 1 + d} \leq \frac{1}{2}$$

and

$$\frac{c + d}{(3c^2 + 1)d^2} \geq \frac{2d}{4d^2} = \frac{1}{2d} \geq \frac{1}{2}.$$

Since $E'(d) \leq 0$, $E(d)$ is decreasing and has the minimum value when d is maximum, hence when $d = c$. Therefore, it suffices to consider this case, when we need to show that $F \geq 0$ for $a \geq b \geq 1 \geq c$ and $ab + 2ac + 2bc + c^2 = 6$, where

$$F = \frac{1}{a} + \frac{1}{b} + \frac{2}{c} + 8 - 3(a + b + 2c).$$

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For fixed a , assume that c and F are functions of b . We have $2(a + b + c)c' + a + 2c = 0$ and

$$F'(b) = \left(-\frac{2}{c^2} - 6\right) - \frac{1}{b^2} - 3 = \left(\frac{1}{c^2} + 3\right) \frac{a + 2c}{a + b + c} - \frac{1}{b^2} - 3.$$

We will show that $F'(b) \geq 0$, that is

$$\frac{\frac{1}{c^2} + 3}{\frac{1}{b^2} + 3} \geq \frac{a + b + c}{a + 2c}, \quad \frac{\frac{1}{c^2} + 3}{\frac{1}{b^2} + 3} - 1 \geq \frac{a + b + c}{a + 2c} - 1, \quad \frac{(b - c)(b + c)}{(3b^2 + 1)c^2} \geq \frac{b - c}{a + 2c}.$$

We need to show that $(b + c)(a + 2c) \geq (3b^2 + 1)c^2$. Since

$$\begin{aligned} (b + c)(a + 2c) - (3b^2 + 1)c^2 &> (b + c)^2 - (3b^2 + 1)c^2 = b(b + 2c - 3bc^2) = \\ &= b^2(1 - c^2) + 2bc(1 - bc), \end{aligned}$$

it suffices to show that $bc \leq 1$. Indeed, from

$$6 = a(b + 2c) + 2bc + c^2 \geq b(b + 2c) + 2bc + c^2 = b^2 + c^2 + 4bc \geq 6bc,$$

we obtain $bc < 1$. Since $F'(b) > 0$, $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. Therefore, it is enough to consider $b = 1$, when we need to show that

$$\frac{1}{a} + \frac{2}{c} - 3 \geq 3(a + 2c - 3)$$

for $a \geq 1 \geq c$ and $a + 2ac + 2c + c^2 = 6$. Since $a = \frac{6-2c-c^2}{2c+1}$, the inequality is equivalent to

$$\frac{(c - 1)^2(c + 4)}{c(6 - 2c - c^2)} \geq \frac{3(c - 1)^2}{2c + 1}, \quad (c - 1)^2(c^3 + 4c^2 + 3c + 4) \geq 0.$$

The proof is finished. The equality occurs for $a = b = c = d = 1$.