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SP.616 Let a, b, c, d be nonnegative real numbers such that at most one of the is less than 1 and $ab + ac + ad + bc + bd + cd = 6$. Prove that:

$$(2a^2 + 1)(2b^2 + 1)(2c^2 + 1)(2d^2 + 1) \geq 81.$$

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Solution by proposer

The equality constraint implies the known inequality

$$a + b + c + d \geq 4.$$

Consider

$$a = \min\{a, b, c, d\} \leq 1, \quad b, c \geq 1, \quad d = \max\{a, b, c, d\} \geq 1,$$

and write the required inequality as $E \geq 0$, where

$$E = \ln(2a^2 + 1) + \ln(2b^2 + 1) + \ln(2c^2 + 1) + \ln(2d^2 + 1) - 4 \ln 3.$$

For fixed c and d , assume that a and E are functions of b . We have

$$(b + c + d)a' + a + c + d = 0 \text{ and}$$

$$\frac{E'(b)}{4} = \frac{b}{2b^2 + 1} + \frac{aa'}{2a^2 + 1} = \frac{b}{2b^2 + 1} - \frac{a(a + c + d)}{(2a^2 + 1)(b + c + d)}.$$

We claim that $E'(b) \geq 0$. Write this inequality as follows:

$$\begin{aligned} \frac{b + c + d}{a + c + d} &\geq \frac{a(2b^2 + 1)}{b(2a^2 + 1)}, \quad \frac{b + c + d}{a + c + d} - 1 \geq \frac{a(2b^2 + 1)}{b(a^2 + 1)} - 1, \\ \frac{b - a}{a + c + d} &\geq \frac{(b - a)(2ab - 1)}{b(2a^2 + 1)}, \quad (b - a)[a + b + c + d - 2ab(c + d)] \geq 0. \end{aligned}$$

We need to show that $a + b + c + d \geq 2ab(c + d)$. Since $a + b + c + d \geq 4$ and

$ab \leq a + b - 1$, it suffices to show that $2 \geq (a + b - 1)(c + d)$, that is

$$2 + c + d \geq (a + b)(c + d), \quad 2 + c + d \geq 6 - a - cd, \quad ab + cd \geq 4 - c - d.$$

Since $4 - c - d \leq a + b$, it suffices to show that $ab + cd \geq a + b$. Indeed,

$$ab + cd - a - b = a(b - 1) + cd - b \geq cd - b \geq d - b \geq 0.$$

From $E'(b) \geq 0$, it follows that $E(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. So, it suffices to prove the inequality $E \geq 0$ for $b = 1$.

Similarly, it suffices to prove the inequality $E \geq 0$ for $c = 1$, hence for $b = c = 1$. We

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need to show that $ad + 2(a + d) = 5$ implies $F \geq 0$, where $F = (2a^2 + 1)(2d^2 + 1) -$

9. We have

$$F = 4a^2d^2 + 2(a + d)^2 - 4ad - 8,$$

hence

$$2F = 8a^2d^2 + (5 - ad)^2 - 8ad - 16 = 9(ad - 1)^2 \geq 0.$$

So, the proof is finished. Equality occurs for $(a, b, c, d) = (1, 1, 1, 1)$.