# *Some refinements of Nesbitt's inequality*

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*In this note, several refinements of Nesbitt's inequality are studied and demonstrated. Simple, double, and multiple refinements are presented.*

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One of the most well-known , used , and cited of the elementary inequalities is *Nesbitt's inequality* , [10] :

- If $a , b , c > 0$ , then , $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2} \quad , \qquad \textbf{(1)}$$

This famous inequality has enjoyed the attention of mathematicians, having over time numerous proofs, extensions, generalizations, and refinements. For this, see for example [1] - [5] .

In what follows, we will highlight a few more refinements of *Nesbitt's inequality*.

Almost all of these refinements were previously published by the author as proposed problems. Therefore – besides the proofs presented below, it would also be useful and rewarding for the reader to check out other solutions published in the magazines or math groups that hosted the proposals .

Thus , in [7] , the following inequality was published, which actually constitutes an initial refinement of *Nesbitt's inequality* :

**1. *Proposition***

If $a , b , c \in \mathbb{R}_{>0}$ , then occur the inequalities ,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq 2\cdot(a+b+c)\cdot\left(\frac{1}{2a+b+c}+\frac{1}{a+2b+c}+\frac{1}{a+b+2c}\right)-3 \geq \frac{3}{2} \cdot \qquad \textbf{(2)}$$

***Proof***

For the first inequality , we have – after a brief preparation , and then by applying *Bergstrom's inequality* three times , we have :

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} =$$

$$=\frac{1}{2}\cdot\left\{\left[\left(\frac{a}{b+c}+1\right)+\left(\frac{b}{c+a}+1\right)-2\right]+\left[\left(\frac{b}{c+a}+1\right)+\left(\frac{c}{a+b}+1\right)-2\right]+\left[\left(\frac{c}{a+b}+1\right)+\left(\frac{a}{b+c}+1\right)-2\right]\right\}=$$

$$=\frac{1}{2}\cdot\left\{\left[(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}\right)-2\right]+\left[(a+b+c)\left(\frac{1}{c+a}+\frac{1}{a+b}\right)-2\right]+\left[(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}\right)-2\right]\right\}\geq$$

$$\overset{\text{Bergström}}{\geq}\frac{1}{2}\cdot\left\{\left[(a+b+c)\cdot\frac{(1+1)^2}{a+b+2c}-2\right]+\left[(a+b+c)\cdot\frac{(1+1)^2}{2a+b+c}-2\right]+\left[(a+b+c)\cdot\frac{(1+1)^2}{a+2b+c}-2\right]\right\}=$$

$$=2\cdot(a+b+c)\cdot\left(\frac{1}{2a+b+c}+\frac{1}{a+2b+c}+\frac{1}{a+b+2c}\right)-3\ .$$

*For the second inequality, we also have - with Bergström's inequality :*

$$2\cdot(a+b+c)\cdot\left(\frac{1}{2a+b+c}+\frac{1}{a+2b+c}+\frac{1}{a+b+2c}\right)-3\overset{\text{Bergström}}{\geq}$$

$$\overset{\text{Bergström}}{\geq}2\cdot(a+b+c)\cdot\frac{(1+1+1)^2}{(2a+b+c)+(a+2b+c)+(a+b+2c)}-3=$$

$$=2\cdot(a+b+c)\cdot\frac{(1+1+1)^2}{4\cdot(a+b+c)}-3=\frac{9}{2}-3=\frac{3}{2}\ .$$

Equality occurs iff $a=b=c$ .

## 2. Proposition

If $a$ , $b$ , $c$ are strictly positive real numbers , then ,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq 3\cdot\left(1-\frac{3}{2}\cdot\frac{ab+bc+ca}{(a+b+c)^2}\right)\geq\frac{3}{2}\ . \tag{3}$$

***Proof***

For the first inequality, after a slight rearrangement and then applying the *inequality of means* , we have :

$$\sum_{cycl}\frac{a}{b+c}=\sum_{cycl}\left(\frac{a}{b+c}+\frac{9}{4}\cdot\frac{a(b+c)}{(a+b+c)^2}\right)-\sum_{cycl}\frac{9}{4}\cdot\frac{a(b+c)}{(a+b+c)^2}\overset{\text{AM-GM}}{\geq}$$

$$\overset{\text{AM-GM}}{\geq}\sum_{cycl}2\cdot\sqrt{\frac{a}{b+c}\cdot\frac{9a(b+c)}{4(a+b+c)^2}}-\frac{9}{4(a+b+c)^2}\cdot\sum_{cycl}a(b+c)=$$

$$=3\cdot\sum_{cycl}\frac{a}{a+b+c}-\frac{9}{2}\cdot\frac{ab+bc+ca}{(a+b+c)^2}=3-\frac{9}{2}\cdot\frac{ab+bc+ca}{(a+b+c)^2}=3\cdot\left(1-\frac{3}{2}\cdot\frac{ab+bc+ca}{(a+b+c)^2}\right)\ .$$

For the second inequality, using the well-known inequality ,

$(a+b+c)^2\geq 3\cdot(ab+bc+ca)$ , we'll have it right away ,

$$-\frac{ab+bc+ca}{(a+b+c)^2}\geq-\frac{1}{3}\ . \tag{*}$$

from which it follows : $$3\cdot\left(1-\frac{3}{2}\cdot\frac{ab+bc+ca}{(a+b+c)^2}\right)\overset{(*)}{\geq}3\cdot\left(1-\frac{3}{2}\cdot\frac{1}{3}\right)=\frac{3}{2}\quad .$$

In [8] another *refinement* ( actually a *double refinement* ) of *Nesbitt's inequality* is published :

**3. *Proposition***

If $a, b, c \in \mathbb{R}_{>0}$ , then occur the inequalities ,

$$\sum_{cycl}\frac{a}{b+c}\overset{①}{\geq}\frac{1}{2}\cdot\sum_{cycl}\frac{a+b}{\sqrt{(b+c)(c+a)}}\overset{②}{\geq}\sum_{cycl}\frac{a+b}{a+b+2c}\overset{③}{\geq}\frac{3}{2}\quad . \qquad (4)$$

***Proof***

For inequality ① , after some slight rearrangements and applications of the *QM-AM inequality* , we successively have :

$$\sum_{cycl}\frac{a}{b+c}=\frac{1}{2}\cdot\sum_{cycl}\left(\frac{a}{b+c}+\frac{b}{c+a}\right)=\frac{1}{2}\cdot\sum_{cycl}\frac{a^2+b^2+c(a+b)}{(b+c)(c+a)}\overset{\text{QM-AM}}{\geq}$$

$$\overset{\text{QM-AM}}{\geq}\frac{1}{2}\cdot\sum_{cycl}\frac{\frac{1}{2}\cdot(a+b)^2+c(a+b)}{(a+c)(b+c)}=\frac{1}{4}\cdot\sum_{cycl}\frac{(a+b)(a+b+2c)}{(a+c)(b+c)}=$$

$$=\frac{1}{4}\cdot\sum_{cycl}\frac{(a+b)\left[(a+c)+(b+c)\right]}{(a+c)(b+c)}\overset{\text{AM-GM}}{\geq}\frac{1}{4}\cdot\sum_{cycl}\frac{(a+b)\cdot 2\cdot\sqrt{(a+c)\cdot(b+c)}}{(a+c)(b+c)}=$$

$$=\frac{1}{2}\cdot\sum_{cycl}\frac{(a+b)}{\sqrt{(a+c)\cdot(b+c)}}\quad .$$

For inequality ② – using the *AM-GM inequality* , we have :

$$\frac{1}{2}\cdot\sum_{cycl}\frac{a+b}{\sqrt{(b+c)(c+a)}}\geq\frac{1}{2}\cdot\sum_{cycl}\frac{a+b}{\frac{(b+c)+(c+a)}{2}}=\sum_{cycl}\frac{a+b}{a+b+2c}\quad .$$

For inequality ③ - after a little preparation to be able to apply *Bergstrom's inequality* and then with the application of the known inequality

$$(a+b+c)^2\geq 3\,(ab+bc+ca)\quad , \qquad (**)$$

and we will successively obtain :

$$:\ \sum_{cycl}\frac{a+b}{a+b+2c}=\sum_{cycl}\frac{(a+b)^2}{(a+b)(a+b+2c)}\overset{\text{Bergström}}{\geq}\frac{\left[\sum_{cycl}(a+b)\right]^2}{\sum_{cycl}(a+b)(a+b+2c)}=$$

$$=\frac{4\cdot\left(\sum_{cycl}a\right)^2}{\sum_{cycl}(a+b)^2+2\cdot\sum_{cycl}(a+b)c}=\frac{4\cdot\left(\sum_{cycl}a\right)^2}{2\cdot\sum_{cycl}a^2+6\cdot\sum_{cycl}ab}=\frac{2\cdot\left(\sum_{cycl}a\right)^2}{\sum_{cycl}a^2+3\cdot\sum_{cycl}ab}=$$

$$=\frac{2\cdot\left(\sum_{cycl}a\right)^2}{\left(\sum_{cycl}a\right)^2+\sum_{cycl}ab}\overset{(**)}{\geq}\frac{2\cdot\left(\sum_{cycl}a\right)^2}{\sum_{cycl}a^2+\frac{\left(\sum_{cycl}a\right)^2}{3}}=\frac{2}{1+\frac{1}{3}}=\frac{3}{2}\ .$$

Equality occurs iff $a=b=c$ .

Also, a double *refinement of Nesbitt's inequality* is proposed in [9] :

**4. *Proposition***

If $a,b,c\in\mathbb{R}_{>0}$ , then occur the following inequalities ,

$$\sum_{cycl}\frac{a}{b+c}\overset{①}{\geq}\frac{1}{3}\cdot\left(\sum_{cycl}a\right)\cdot\left(\sum_{cycl}\frac{1}{a+b}\right)\overset{②}{\geq}\frac{2}{3}\cdot\left(\sum_{cycl}a\right)\cdot\left(\sum_{cycl}\frac{1}{2a+b+c}\right)\overset{③}{\geq}\frac{3}{2}\ . \qquad (5)$$

***Proof***

① Let's note that – without reducing generality, we have the implication

$$a\geq b\geq c\ \Rightarrow\ \frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}\ ,\ \text{so the triplets}\ (a,b,c)\ ,\ \left(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}\right)$$

they are *similarly ordered* , so with *Chebyshev's inequality* , we have :

$$\sum_{cycl}\frac{a}{b+c}=\sum_{cycl}a\cdot\frac{1}{b+c}\overset{Chebyshev}{\geq}\frac{1}{3}\cdot\left(\sum_{cycl}a\right)\cdot\left(\sum_{cycl}\frac{1}{b+c}\right)=\frac{1}{3}\cdot\left(\sum_{cycl}a\right)\cdot\left(\sum_{cycl}\frac{1}{a+b}\right)\ .$$

For inequality ② – using the *Bergstrom's inequality* , we have :

$$\frac{1}{3}\cdot\left(\sum_{cycl}a\right)\cdot\left(\sum_{cycl}\frac{1}{a+b}\right)=\frac{1}{6}\cdot\left(\sum_{cycl}a\right)\cdot\sum_{cycl}\left(\frac{1}{a+b}+\frac{1}{b+c}\right)\overset{Bergström}{\geq}\frac{1}{6}\cdot\left(\sum_{cycl}a\right)\cdot\sum_{cycl}\frac{(1+1)^2}{(a+b)+(b+c)}=$$

$$=\frac{2}{3}\cdot\left(\sum_{cycl}a\right)\cdot\left(\sum_{cycl}\frac{1}{a+2b+c}\right)=\frac{2}{3}\cdot\left(\sum_{cycl}a\right)\cdot\left(\sum_{cycl}\frac{1}{2a+b+c}\right)\ .$$

For inequality ③ – also with the application of *Bergström's inequality* , we have:

$$\frac{2}{3}\cdot\left(\sum_{cycl} a\right)\cdot\left(\sum_{cycl}\frac{1}{2a+b+c}\right)\overset{Bergström}{\geq}\frac{2}{3}\cdot\left(\sum_{cycl} a\right)\cdot\frac{3^2}{\sum_{cycl}(2a+b+c)}=\frac{2}{3}\cdot\left(\sum_{cycl} a\right)\cdot\frac{3^2}{4\cdot\left(\sum_{cycl} a\right)}=\frac{3}{2}\ .$$

Equality occurs iff $a = b = c$ .

**5. *Remark***

In *Propositions* 1 and 2 – we dealt with *simple refinements* of *Nesbitt's inequality* (that is, only a single expression was inserted between the sides of inequality (1)) whereas in *Propositions* 3 and 4 – we have *double refinements* (that is, two intermediate expressions were inserted between the sides of *Nesbitt's inequality*). We can have *multiple refinements* – with any number of intermediate expressions between the terms of *Nesbitt's inequality* , even infinitely many, as in :

**6. *Proposition*** ( ***a multiple refinement of Nesbitt's inequality***, **[2] , [5]** )

If $a , b , c$ are strictly positive real numbers , then ,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{\sqrt{b}}{\sqrt{c}+\sqrt{a}}+\frac{\sqrt{c}}{\sqrt{a}+\sqrt{b}}\geq$$

$$\geq\frac{\sqrt[3]{a}}{\sqrt[3]{b}+\sqrt[3]{c}}+\frac{\sqrt[3]{b}}{\sqrt[3]{c}+\sqrt[3]{a}}+\frac{\sqrt[3]{c}}{\sqrt[3]{a}+\sqrt[3]{b}}\geq\ldots\geq$$

$$\geq\frac{\sqrt[n]{a}}{\sqrt[n]{b}+\sqrt[n]{c}}+\frac{\sqrt[n]{b}}{\sqrt[n]{c}+\sqrt[n]{a}}+\frac{\sqrt[n]{c}}{\sqrt[n]{a}+\sqrt[n]{b}}\geq\ldots\geq\frac{3}{2}\ . \qquad \textbf{(6)}$$

**5. *Remark***

In the *double* or *multiple refinements* mentioned above - that is , in relations (4) , (5) , (6) , we also have an ordering of the expressions that achieve the *refinement* of the inequality . It would be very interesting to compare all the expressions that achieve the refinement of *Nesbitt's inequality* among themselves . It's a challenge we throw out to lovers of inequalities .

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