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4101. In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{r_c}+\frac{b^4+c^4}{r_a}+\frac{c^4+a^4}{r_b}\ge 288r^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$r_ar_br_c=\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}=\frac{F^3s}{s(s-a)(s-b)(s-c)}=\frac{F^3s}{F^2}=Fs$$

$$\frac{a^4+b^4}{r_c}+\frac{b^4+c^4}{r_a}+\frac{c^4+a^4}{r_b}=\sum_{cyc}\frac{a^4+b^4}{r_c}\overset{AM-GM}{\ge}$$

$$\ge 2\sum_{cyc}\frac{a^2b^2}{r_c}\overset{AM-GM}{\ge}2\cdot 3\cdot\sqrt[3]{\frac{(abc)^4}{r_ar_br_c}}=6abc\cdot\sqrt[3]{\frac{abc}{Fs}}=6\cdot 4RF\cdot\sqrt[3]{\frac{4RF}{Fs}}=$$

$$=24RF\sqrt[3]{\frac{4R}{s}}\overset{MITRINOVIC}{\ge}24RF\cdot\sqrt[3]{\frac{4}{s}\cdot\frac{2}{3\sqrt{3}}s}=$$

$$=24Rrs\cdot\frac{2}{\sqrt{3}}\overset{MITRINOVIC}{\ge}24R\cdot 3\sqrt{3}r\cdot\frac{2}{\sqrt{3}}=24\cdot 6Rr\overset{EULER}{\ge}144\cdot 2r\cdot r=288r^3$$

Equality holds for $a=b=c$.

4102. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{\sqrt{r_a}}\le\sum_{cyc}\frac{a}{\sqrt{h_a}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Jenish Rijal-Nepal

$$WLOG\ let's\ assume\ that\ a\ge b\ge c\ \Rightarrow\frac{1}{\sqrt{r_a}}\le\frac{1}{\sqrt{r_b}}\le\frac{1}{\sqrt{r_c}}\ and\ \frac{1}{\sqrt{h_a}}\ge\frac{1}{\sqrt{h_b}}\ge\frac{1}{\sqrt{h_c}}.$$

$$\frac{1}{\sqrt{h_a}}=\sqrt{\frac{\frac{1}{r_b}+\frac{1}{r_c}}{2}}\overset{Power\ Mean}{\ge}\frac{1}{2}\left(\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\right)\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{r_a}}\le\sum_{cyc}\frac{1}{\sqrt{h_a}}$$

$$\sum_{cyc}\left(a\cdot\frac{1}{\sqrt{r_a}}\right)\overset{Chebyshev}{\le}\frac{1}{3}\cdot\sum_{cyc}(a)\cdot\sum_{cyc}\left(\frac{1}{\sqrt{r_a}}\right)\le\frac{1}{3}\cdot\sum_{cyc}(a)\cdot\sum_{cyc}\left(\frac{1}{\sqrt{h_a}}\right)$$

$$\le\sum_{cyc}\left(a\cdot\frac{1}{\sqrt{h_a}}\right)$$

$$\therefore \sum_{cyc} \frac{a}{\sqrt{r_a}} \le \sum_{cyc} \frac{a}{\sqrt{h_a}}$$

Equality holds if and only if the triangle is equilateral.

4103. In ΔABC the following relationship holds:

$$\sum_{cyc} \sqrt{\frac{a^3}{r_a}} \le \sum_{cyc} \sqrt{\frac{a^3}{h_a}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Jenish Rijal-Nepal

$$WLOG\ let's\ assume\ that\ a \ge b \ge c\ \Rightarrow$$

$$\sqrt{a^3} \ge \sqrt{b^3} \ge \sqrt{c^3} \Rightarrow \frac{1}{\sqrt{r_a}} \le \frac{1}{\sqrt{r_b}} \le \frac{1}{\sqrt{r_c}}\ and\ \frac{1}{\sqrt{h_a}} \ge \frac{1}{\sqrt{h_b}} \ge \frac{1}{\sqrt{h_c}}.$$

$$\frac{1}{\sqrt{h_a}} = \sqrt{\frac{\frac{1}{r_b} + \frac{1}{r_c}}{2}} \overset{Power\ Mean}{\overbrace{\ge}} \frac{1}{2}\left(\frac{1}{\sqrt{r_b}} + \frac{1}{\sqrt{r_c}}\right) \Leftrightarrow \sum_{cyc} \frac{1}{\sqrt{r_a}} \le \sum_{cyc} \frac{1}{\sqrt{h_a}}$$

$$\sum_{cyc} \left(\sqrt{a^3} \cdot \frac{1}{\sqrt{r_a}}\right) \overset{Chebyshev}{\overbrace{\le}} \frac{1}{3} \cdot \sum_{cyc} \left(\sqrt{a^3}\right) \cdot \sum_{cyc} \left(\frac{1}{\sqrt{r_a}}\right) \le$$

$$\le \frac{1}{3} \cdot \sum_{cyc} \left(\sqrt{a^3}\right) \cdot \sum_{cyc} \left(\frac{1}{\sqrt{h_a}}\right) \le \sum_{cyc} \left(\sqrt{a^3} \cdot \frac{1}{\sqrt{h_a}}\right)$$

$$\therefore \sum_{cyc} \sqrt{\frac{a^3}{r_a}} \le \sum_{cyc} \sqrt{\frac{a^3}{h_a}}$$

Equality holds if and only if the triangle is equilateral.

4104. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a^2}{\sqrt{r_a}} \le \sum_{cyc} \frac{a^2}{\sqrt{h_a}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Jenish Rijal-Nepal

$$WLOG\ let's\ assume\ that\ a \ge b \ge c\ \Rightarrow$$

$$a^2 \ge b^2 \ge c^2\ \Rightarrow \frac{1}{\sqrt{r_a}} \le \frac{1}{\sqrt{r_b}} \le \frac{1}{\sqrt{r_c}}\ and\ \frac{1}{\sqrt{h_a}} \ge \frac{1}{\sqrt{h_b}} \ge \frac{1}{\sqrt{h_c}}$$

$$\frac{1}{\sqrt{h_a}}=\sqrt{\frac{\frac{1}{r_b}+\frac{1}{r_c}}{2}}\overset{\text{Power Mean}}{\overbrace{\geq}}\frac{1}{2}\left(\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\right)\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{r_a}}\leq\sum_{cyc}\frac{1}{\sqrt{h_a}}$$

$$\sum_{cyc}\left(a^2\cdot\frac{1}{\sqrt{r_a}}\right)\overset{\text{Chebyshev}}{\overbrace{\leq}}\frac{1}{3}\cdot\sum_{cyc}(a^2)\cdot\sum_{cyc}\left(\frac{1}{\sqrt{r_a}}\right)\leq$$

$$\leq\frac{1}{3}\cdot\sum_{cyc}(a^2)\cdot\sum_{cyc}\left(\frac{1}{\sqrt{h_a}}\right)\leq\sum_{cyc}\left(a^2\cdot\frac{1}{\sqrt{h_a}}\right)$$

$$\therefore\sum_{cyc}\frac{a^2}{\sqrt{r_a}}\leq\sum_{cyc}\frac{a^2}{\sqrt{h_a}}$$

Equality holds if and only if the triangle is equilateral.

4105. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{a}}{r_a}\leq\sum_{cyc}\frac{\sqrt{a}}{h_a}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Jenish Rijal-Nepal

$WLOG\ let's\ assume\ that\ a\geq b\geq c\ \Rightarrow\sqrt{a}\geq\sqrt{b}\geq\sqrt{c}\ \Rightarrow$

$$\frac{1}{r_a}\leq\frac{1}{r_b}\leq\frac{1}{r_c}\ \text{and}\ \frac{1}{h_a}\geq\frac{1}{h_b}\geq\frac{1}{h_c}.$$

$$\sum_{cyc}\left(\sqrt{a}\cdot\frac{1}{r_a}\right)\overset{\text{Chebyshev}}{\overbrace{\leq}}\frac{1}{3}\cdot\sum_{cyc}(\sqrt{a})\cdot\sum_{cyc}\left(\frac{1}{r_a}\right)\overset{\sum\frac{1}{r_a}=\sum\frac{1}{h_a}}{\overbrace{\leq}}$$

$$\leq\frac{1}{3}\cdot\sum_{cyc}(\sqrt{a})\cdot\sum_{cyc}\left(\frac{1}{h_a}\right)\overset{\text{Chebyshev}}{\overbrace{\leq}}\sum_{cyc}\left(\sqrt{a}\cdot\frac{1}{h_a}\right)$$

$$\therefore\sum_{cyc}\frac{\sqrt{a}}{r_a}\leq\sum_{cyc}\frac{\sqrt{a}}{h_a}$$

Equality holds if and only if the triangle is equilateral.

4106. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sin^2 B+\sin^2 C}{(\sin A\sin B)(\sin A\sin C+\sin^2 B)}\geq 4$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS} \overset{\text{CBS}}{\geq} \sum_{\text{cyc}} \frac{\sin^2 B + \sin^2 C}{(\sin A \sin B)\left(\sqrt{(\sin^2 A + \sin^2 B)(\sin^2 B + \sin^2 C)}\right)} \overset{\text{AM-GM}}{\geq}$$

$$\frac{3}{\left(\sqrt[3]{\sin A \sin B \sin C}\right)^2} \overset{\text{AM-GM}}{\geq} \frac{3}{\left(\frac{\sum_{\text{cyc}} \sin A}{3}\right)^2} = \frac{27R^2}{s^2} \overset{\text{Mitrinovic}}{\geq} 4\ \forall\ \Delta\ ABC,$$

$$'' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

4107. **In ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{r_b\sqrt{r_c}}{r_a{}^2} \geq \sqrt{\frac{3}{r}}$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

$$\text{Let } r_a = \frac{1}{x}, r_b = \frac{1}{y}, \text{ and } r_c = \frac{1}{z}.$$

$$\text{Using the identity } \sum_{cyc} \frac{1}{r_a} = \frac{1}{r}, \text{ we have } x + y + z = \frac{1}{r}.$$

$$\text{Substituting these into the LHS: } LHS = \sum_{cyc} \frac{r_b\sqrt{r_c}}{r_a{}^2} = \sum_{cyc} \frac{\frac{1}{y}\sqrt{\frac{1}{z}}}{\frac{1}{x^2}} = \sum_{cyc} \frac{x^2}{y\sqrt{z}}$$

$$\text{Now, } \sum_{cyc} \frac{x^2}{y\sqrt{z}} \overset{BERGSTROM}{\geq} \frac{(x+y+z)^2}{\sum_{cyc} y\sqrt{z}} \mapsto (1)$$

$$\text{Again by Cauchy's Inequality,}$$

$$\left(\sum_{cyc} \sqrt{y} \cdot \sqrt{yz}\right)^2 = \left(\sum_{cyc} y\sqrt{z}\right)^2 \leq (x+y+z)(xy+yz+zx) \Rightarrow$$

$$\sum_{cyc} y\sqrt{z} \leq \sqrt{(x+y+z)(xy+yz+zx)}$$

$$\text{Substituting this bound back into (1):}$$

$$\sum_{cyc} \frac{x^2}{y\sqrt{z}} \geq \frac{(x+y+z)^2}{\sqrt{(x+y+z)(xy+yz+zx)}} = \sqrt{\frac{(x+y+z)^3}{(xy+yz+zx)}}$$

Since $(a+b+c)^2 \geq 3(ab+bc+ac)$, *it follows that*:

$$\sum_{cyc} \frac{h_b\sqrt{h_c}}{h_a{}^2} = \sum_{cyc} \frac{x^2}{y\sqrt{z}} \geq \sqrt{\frac{(x+y+z)^3}{(xy+yz+zx)}} \geq \sqrt{3(x+y+z)} = \sqrt{3 \cdot \frac{1}{r}}$$

$$\therefore \sum_{cyc} \frac{r_b\sqrt{r_c}}{r_a{}^2} \geq \sqrt{\frac{3}{r}}$$

Equality holds if and only if the triangle is equilateral.

4108. In ΔABC the following relationship holds:

$$\sum_{cyc} \left(\frac{1}{r_a} + \frac{1}{r_b}\right)\left(\sqrt{r_a} + \sqrt{r_b}\right)^2 \leq \frac{12R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$WLOG: a \leq b \leq c\ ;\ r_a \leq r_b \leq r_c;\ \ \sqrt{r_a} \leq \sqrt{r_b} \leq \sqrt{r_c}\ ;\ \frac{1}{r_a} \geq \frac{1}{r_b} \geq \frac{1}{r_c}$$

$$\sum_{cyc} \left(\frac{1}{r_a} + \frac{1}{r_b}\right)\left(\sqrt{r_a} + \sqrt{r_b}\right)^2 \overset{Chebyshev}{\leq} \frac{1}{3}\sum_{cyc}\left(\frac{1}{r_a} + \frac{1}{r_b}\right) \cdot \sum_{cyc}\left(\sqrt{r_a} + \sqrt{r_b}\right)^2 =$$

$$= \frac{1}{3} \cdot 2\left(\sum_{cyc} \frac{1}{r_a}\right) \cdot 2\left(\sum_{cyc} r_a + \sum_{cyc} \sqrt{r_a \cdot r_b}\right) \overset{A-G}{\leq} \frac{4}{3r}\left((4R+r) + (4R+r)\right) \overset{Euler}{\leq}$$

$$\leq \frac{4}{3r} \cdot \frac{9R}{2} \cdot 2 = \frac{12R}{r}$$

Equality holds for: $a = b = c$

4109.

In any Δ ABC the following relationship holds :

$$\left(\frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c}\right)^2 \geq \frac{r_a + r_b + r_c}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{?}{\geq} \frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right)=3+\sum_{cyc}\frac{x+y}{z}$$

$$=3+\sum_{cyc}\frac{y}{x}+\sum_{cyc}\frac{x}{y} \Leftrightarrow \sum_{cyc}\frac{x^2}{y^2}+\sum_{cyc}\frac{y}{x} \underset{①}{\overset{?}{\geq}} 3+\sum_{cyc}\frac{x}{y}$$

$$\text{Now, } \sum_{cyc}\frac{x^2}{y^2} \geq \frac{1}{3}\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{AM-GM}{\geq} \frac{1}{3}.3.\sum_{cyc}\frac{x}{y} \Rightarrow \sum_{cyc}\frac{x^2}{y^2} \geq \sum_{cyc}\frac{x}{y} \text{ and } \because \sum_{cyc}\frac{y}{x} \overset{AM-GM}{\geq} 3$$

$$\therefore \sum_{cyc}\frac{x^2}{y^2}+\sum_{cyc}\frac{y}{x} \geq \sum_{cyc}\frac{x}{y}+3 \Rightarrow ① \text{ is true}$$

$$\therefore \left(\sum_{cyc}\frac{x}{y}\right)^2 \geq \frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right)=\frac{1}{xyz}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right) \text{ and with}$$

$$x\equiv r_b, y\equiv r_a, z\equiv r_c, \left(\frac{r_b}{r_a}+\frac{r_a}{r_c}+\frac{r_c}{r_b}\right)^2 \geq \frac{1}{r_ar_br_c}\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)$$

$$=\frac{1}{s^2r}\left(\sum_{cyc}r_a\right).s^2=\frac{r_a+r_b+r_c}{r} \text{ and so, } \left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)^2 \geq \frac{r_a+r_b+r_c}{r}$$

$$\forall \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4110. In ΔABC the following relationship holds:

$$\frac{27r}{s} \leq \sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A} \leq \frac{27R^2}{4rs}$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

$$\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}=\sum_{cyc}\frac{\frac{s(s-b)}{ac}+\frac{s(s-c)}{ab}}{\frac{2rs}{bc}}=$$

$$=\sum_{cyc}\frac{s(s-b)\cdot b+s(s-c)\cdot c}{2ars}=\sum_{cyc}\frac{b(s-b)+c(s-c)}{2ar}$$

$$=\sum_{cyc}\frac{b(a+c-b)+c(a+b-c)}{4ar}=$$

$$=\sum_{cyc}\frac{ab+ca+2bc-(b^2+c^2)}{4ar}=\sum_{cyc}\frac{a(b+c)-(b-c)^2}{4ar}=$$

$$=\sum_{cyc}\left[\frac{(b+c)}{4r}-\frac{(b-c)^2}{4ar}\right]=\frac{s}{r}-\sum_{cyc}\frac{(b-c)^2}{4ar}\le\frac{s}{r}\quad\because\sum_{cyc}\frac{(b-c)^2}{4ar}\ge 0$$

$$\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\le\frac{s}{r}=\frac{(2s)^2}{4rs}\overset{Mitrinovic}{\le}\frac{\left(3\sqrt{3}R\right)^2}{4rs}=\frac{27R^2}{4rs}$$

$$\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\overset{AM-GM}{\ge}\sum_{cyc}\frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{\sin A}=$$

$$=\sum_{cyc}\frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{8\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{8\prod\left(\sin\frac{A}{2}\cos\frac{A}{2}\right)}}=$$

$$3\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}=3\sqrt[3]{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}\overset{Jensen}{\ge}$$

$$\ge 3\sqrt[3]{\cot\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)}=3\sqrt[3]{3\sqrt{3}}=3\sqrt{3}=\frac{27}{3\sqrt{3}}$$

$$\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\ge\frac{27}{3\sqrt{3}}=27\cdot\frac{1}{3\sqrt{3}}\overset{Mitrinovic}{\ge}27\cdot\frac{r}{s}=\frac{27r}{s}$$

$$\therefore\ \frac{27r}{s}\le\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\le\frac{27R^2}{4rs}$$

Equality holds if and only if the triangle is equilateral.

4111. In any Δ ABC the following relationship holds :

$$\left(\frac{h_b}{h_a}+\frac{h_c}{h_b}+\frac{h_a}{h_c}\right)^2\ge\frac{h_a+h_b+h_c}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{x}{y}\right)^2\overset{?}{\ge}\frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right)=3+\sum_{cyc}\frac{x+y}{z}$$

$$= 3 + \sum_{cyc}\frac{y}{x} + \sum_{cyc}\frac{x}{y} \Leftrightarrow \sum_{cyc}\frac{x^2}{y^2} + \sum_{cyc}\frac{y}{x} \underset{①}{\overset{?}{\geq}} 3 + \sum_{cyc}\frac{x}{y}$$

$$\text{Now, } \sum_{cyc}\frac{x^2}{y^2} \geq \frac{1}{3}\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{AM-GM}{\geq} \frac{1}{3}.3.\sum_{cyc}\frac{x}{y} \Rightarrow \sum_{cyc}\frac{x^2}{y^2} \geq \sum_{cyc}\frac{x}{y} \text{ and } \because \sum_{cyc}\frac{y}{x} \overset{AM-GM}{\geq} 3$$

$$\therefore \sum_{cyc}\frac{x^2}{y^2} + \sum_{cyc}\frac{y}{x} \geq \sum_{cyc}\frac{x}{y} + 3 \Rightarrow ① \text{ is true}$$

$$\therefore \left(\sum_{cyc}\frac{x}{y}\right)^2 \geq \frac{1}{xyz}\left(3xyz + \sum_{cyc}(xy(x+y))\right) = \frac{1}{xyz}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right) \text{ and with}$$

$$x \equiv h_b, y \equiv h_a, z \equiv h_c, \left(\frac{h_b}{h_a} + \frac{h_a}{h_c} + \frac{h_c}{h_b}\right)^2 \geq \frac{1}{h_a h_b h_c}\left(\sum_{cyc}h_a\right)\left(\sum_{cyc}h_a h_b\right)$$

$$= \frac{1}{s^2 r}\left(\sum_{cyc}h_a\right).s^2 = \frac{h_a + h_b + h_c}{r} \text{ and so, } \left(\frac{h_b}{h_a} + \frac{h_c}{h_b} + \frac{h_a}{h_c}\right)^2 \geq \frac{h_a + h_b + h_c}{r}$$

$$\forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4112.

In any Δ ABC the following relationship holds

$$\frac{s}{r} + \sqrt{2\left(\frac{2R - r}{r} + \sum_{cyc}\frac{l_b + lc}{h_a}\right)} \geq$$

$$\frac{(n_a + n_b + n_c)^2}{s(h_a + h_b + h_c) - \left(h_a.\sqrt{2r_a h_a} + h_b.\sqrt{2r_b h_b} + h_c.\sqrt{2r_c h_c}\right)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$s(h_a + h_b + h_c) - \left(h_a.\sqrt{2r_a h_a} + h_b.\sqrt{2r_b h_b} + h_c.\sqrt{2r_c h_c}\right)$$

$$=\sum_{cyc}\frac{h_a(s^2-2r_ah_a)}{s+\sqrt{2r_ah_a}}\overset{\text{Bogdan Fustei}}{=}\sum_{cyc}\frac{n_a^2}{\frac{s}{h_a}+\sqrt{\frac{2r_a}{h_a}}}\overset{\text{Bergstrom}}{\geq}\frac{(n_a+n_b+n_c)^2}{\frac{s}{r}+\sum_{cyc}\sqrt{\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}}$$

$$=\frac{(n_a+n_b+n_c)^2}{\frac{s}{r}+\sum_{cyc}\sqrt{\frac{\sin^2\frac{A}{2}}{\frac{r}{4R}}}}\Rightarrow s(h_a+h_b+h_c)-\left(h_a.\sqrt{2r_ah_a}+h_b.\sqrt{2r_bh_b}+h_c.\sqrt{2r_ch_c}\right)$$

$$\overset{(1)}{\geq}\frac{(n_a+n_b+n_c)^2}{\frac{s}{r}+\sqrt{\frac{4R}{r}}.\left(\sum_{cyc}\sin\frac{A}{2}\right)}\text{ and again, }\sum_{cyc}\frac{l_b+lc}{h_a}=\sum_{cyc}\left(l_a\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\right)$$

$$=\sum_{cyc}\left(\frac{2bc}{b+c}.\cos\frac{A}{2}.\left(\frac{b+c}{2rs}\right)\right)=\frac{4Rrs}{rs}.\sum_{cyc}\frac{\cos\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}=\sum_{cyc}\operatorname{cosec}\frac{A}{2}$$

$$\Rightarrow\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\frac{l_b+lc}{h_a}\right)}\overset{(2)}{=}\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)}$$

Now, (1) and (2)

$$\Rightarrow\left(\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\frac{l_b+lc}{h_a}\right)}\right)\left(\begin{matrix}s(h_a+h_b+h_c)-\\(h_a.\sqrt{2r_ah_a}+h_b.\sqrt{2r_bh_b}+h_c.\sqrt{2r_ch_c})\end{matrix}\right)\geq$$

$$\left(\frac{(n_a+n_b+n_c)^2}{\frac{s}{r}+\sqrt{\frac{4R}{r}}.\left(\sum_{cyc}\sin\frac{A}{2}\right)}\right).\left(\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)}\right)\overset{?}{\geq}(n_a+n_b+n_c)^2$$

$$\Leftrightarrow 2\left(\frac{2R-r}{r}+\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)\overset{?}{=}\frac{4R}{r}\left(\frac{2R-r}{2R}+2.\frac{r}{4R}.\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)\rightarrow\text{true}$$

$$\therefore\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum_{cyc}\frac{l_b+lc}{h_a}\right)}\geq$$

$$\frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a.\sqrt{2r_ah_a}+h_b.\sqrt{2r_bh_b}+h_c.\sqrt{2r_ch_c}\right)}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equil*ateral* (QED)

4113. In any Δ ABC the following relationship holds :

$$\max\{a,b,c\} \ge 2s.\frac{R-r}{2R-r} \ge \frac{3abc}{ab+bc+ca} \ge \min\{a,b,c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x,$

$c = x+y$ & $s = x+y+z$ and furthermore, for proving $\max\{a,b,c\} \ge 2s.\frac{R-r}{2R-r}$,

we assume WLOG $a = \max\{a,b,c\} \therefore a \ge b,c \Rightarrow a \ge b,c \Rightarrow y+z \ge z+x, x+y$

$$\Rightarrow \boxed{y,z \ge x} \text{ and then : } \max\{a,b,c\} \overset{?}{\ge} 2s.\frac{R-r}{2R-r}$$

$$\Leftrightarrow y+z \overset{?}{\ge} \frac{(x+y+z)\big((y+z)(z+x)(x+y)-4xyz\big)}{(y+z)(z+x)(x+y)-2xyz}$$

$$\Leftrightarrow (x^2yz - x^3y) + (x^2yz - x^3z) + (xy^2z - x^2y^2) + (xyz^2 - x^2z^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow x^2y(z-x) + x^2z(y-x) + xy^2(z-x) + xz^2(y-x) \overset{?}{\ge} 0 \to \text{true} \because y,z \ge x$$

$$\therefore \boxed{\max\{a,b,c\} \ge 2s.\frac{R-r}{2R-r}} \text{ and now, for proving } \frac{3abc}{ab+bc+ca} \overset{?}{\ge} \min\{a,b,c\},$$

we assume WLOG $a = \min\{a,b,c\} \therefore a \le b,c \Rightarrow y+z \le z+x, x+y$

$$\Rightarrow \boxed{y,z \le x} \text{ and then : } \frac{3abc}{ab+bc+ca} \overset{?}{\ge} \min\{a,b,c\}$$

$$\Leftrightarrow \frac{3(y+z)(z+x)(x+y)}{(y+z)(z+x)+(z+x)(x+y)+(x+y)(y+z)} \overset{?}{\ge} y+z \Leftrightarrow 2x^2 \overset{?}{\ge} y^2+z^2$$

$$\Leftrightarrow (x^2-y^2)+(x^2-z^2) \overset{?}{\ge} 0 \to \text{true} \because x \ge y,z \therefore \boxed{\frac{3abc}{ab+bc+ca} \ge \min\{a,b,c\}}$$

$$\text{and finally, } \frac{3abc}{ab+bc+ca} \overset{\text{Gerretsen}}{\le} \frac{12Rrs}{20Rr-4r^2} \overset{?}{\le} 2s.\frac{R-r}{2R-r}$$

$$\Leftrightarrow (4R-r)(R-2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \therefore \boxed{2s.\frac{R-r}{2R-r} \ge \frac{3abc}{ab+bc+ca}} \text{ and so,}$$

$$\max\{a,b,c\} \ge 2s.\frac{R-r}{2R-r} \ge \frac{3abc}{ab+bc+ca} \ge \min\{a,b,c\}\ \forall \Delta \text{ ABC},$$

$'' ='' $ iff Δ ABC is equilateral (QED)

4114. In any Δ ABC the following relationship holds :

$$\frac{h_a}{r_a+r_b}+\frac{h_b}{r_b+r_c}+\frac{h_c}{r_c+r_a}\le\frac{2R^2+r^2}{R(R+r)}$$

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$$\frac{h_a}{r_a+r_b}+\frac{h_b}{r_b+r_c}+\frac{h_c}{r_c+r_a}=$$

$$=\frac{1}{\prod_{cyc}\left(4R\cos^2\frac{A}{2}\right)}\cdot\sum_{cyc}\frac{2rs.\left(4R\cos^2\frac{A}{2}\right)\left(4R\cos^2\frac{B}{2}\right)}{4R\left(\cos^2\frac{A}{2}\right)\left(\tan\frac{A}{2}\right)}=$$

$$=\frac{4R}{64R^3.\frac{s^2}{16R^2}}\cdot\sum_{cyc}\left(\frac{2r_ar_br_c}{r_a}\cdot\frac{sb(s-b)}{abc}\right)=\frac{2s^2}{abcs^2}\sum_{cyc}\left(b(s-a)(s-b)\right)=$$

$$\frac{2}{(y+z)(z+x)(x+y)}\cdot\sum_{cyc}(xy(z+x))\ \left(\begin{matrix}x=s-a,y=s-b,z=s-c\text{ and then :}\\ a=y+z,b=z+x,c=x+y\end{matrix}\right)$$

$$\overset{?}{\le}\frac{2R^2+r^2}{R(R+r)}=\frac{\frac{2R^2}{r^2}+1}{\frac{R^2}{r^2}+\frac{R}{r}}=\frac{\frac{(y+z)^2(z+x)^2(x+y)^2}{8x^2y^2z^2}+1}{\frac{(y+z)^2(z+x)^2(x+y)^2}{16x^2y^2z^2}+\frac{(y+z)(z+x)(x+y)}{4xyz}}$$

$$\Leftrightarrow xyz\sum_{cyc}x^3+\sum_{cyc}x^2y^4+xyz\sum_{cyc}xy^2\underset{(*)}{\overset{?}{\ge}}3xyz\sum_{cyc}x^2y$$

Now, $m^2+n^2+p^2\ge mn+np+pm$ and choosing $m\equiv xy^2,n\equiv yz^2,p\equiv zx^2$,

we get : $\sum_{cyc}x^2y^4\ge xyz\sum_{cyc}x^2y$ and since : $\sum_{cyc}x^3+\sum_{cyc}xy^2\overset{AM-GM}{\ge}2\sum_{cyc}x^2y$

$$\therefore\text{ LHS of }(*)\ge 2xyz\sum_{cyc}x^2y+xyz\sum_{cyc}x^2y\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{h_a}{r_a+r_b}+\frac{h_b}{r_b+r_c}+\frac{h_c}{r_c+r_a}\le\frac{2R^2+r^2}{R(R+r)}\ \forall\Delta\text{ ABC},$$

'' ='' iff Δ ABC is equilateral (QED)

4115. In any Δ ABC the following relationship holds :

$$\max\{a,b,c\}\ge 2s.\frac{R-r}{2R-r}\ge\frac{3abc}{ab+bc+ca}\ge\min\{a,b,c\}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = s - a, y = s - b, z = s - c \text{ and then : } a = y + z, b = z + x,$$
$$c = x + y \;\&\; s = x + y + z \text{ and furthermore, for proving } \max\{a, b, c\} \geq 2s.\frac{R-r}{2R-r},$$
$$\text{we assume WLOG } a = \max\{a, b, c\} \therefore a \geq b, c \Rightarrow a \geq b, c \Rightarrow y + z \geq z + x, x + y$$
$$\Rightarrow \boxed{y, z \geq x} \text{ and then : } \max\{a, b, c\} \overset{?}{\geq} 2s.\frac{R-r}{2R-r}$$
$$\Leftrightarrow y + z \overset{?}{\geq} \frac{(x+y+z)\big((y+z)(z+x)(x+y) - 4xyz\big)}{(y+z)(z+x)(x+y) - 2xyz}$$
$$\Leftrightarrow (x^2yz - x^3y) + (x^2yz - x^3z) + (xy^2z - x^2y^2) + (xyz^2 - x^2z^2) \overset{?}{\geq} 0$$
$$\Leftrightarrow x^2y(z-x) + x^2z(y-x) + xy^2(z-x) + xz^2(y-x) \overset{?}{\geq} 0 \to \text{true} \because y, z \geq x$$
$$\therefore \boxed{\max\{a, b, c\} \geq 2s.\frac{R-r}{2R-r}} \text{ and now, for proving } \frac{3abc}{ab+bc+ca} \overset{?}{\geq} \min\{a, b, c\},$$
$$\text{we assume WLOG } a = \min\{a, b, c\} \therefore a \leq b, c \Rightarrow y + z \leq z + x, x + y$$
$$\Rightarrow \boxed{y, z \leq x} \text{ and then : } \frac{3abc}{ab+bc+ca} \overset{?}{\geq} \min\{a, b, c\}$$
$$\Leftrightarrow \frac{3(y+z)(z+x)(x+y)}{(y+z)(z+x) + (z+x)(x+y) + (x+y)(y+z)} \overset{?}{\geq} y + z \Leftrightarrow 2x^2 \overset{?}{\geq} y^2 + z^2$$
$$\Leftrightarrow (x^2 - y^2) + (x^2 - z^2) \overset{?}{\geq} 0 \to \text{true} \because x \geq y, z \therefore \boxed{\frac{3abc}{ab+bc+ca} \geq \min\{a, b, c\}}$$
$$\text{and finally, } \frac{3abc}{ab+bc+ca} \overset{\text{Gerretsen}}{\leq} \frac{12Rrs}{20Rr - 4r^2} \overset{?}{\leq} 2s.\frac{R-r}{2R-r}$$
$$\Leftrightarrow (4R - r)(R - 2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \therefore \boxed{2s.\frac{R-r}{2R-r} \geq \frac{3abc}{ab+bc+ca}} \text{ and so,}$$
$$\max\{a, b, c\} \geq 2s.\frac{R-r}{2R-r} \geq \frac{3abc}{ab+bc+ca} \geq \min\{a, b, c\} \;\forall \Delta ABC,$$
$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4116. In any Δ ABC the following relationship holds :

$$\max\{r_a, r_b, r_c\} \geq \sqrt{4R^2 - 7r^2} \geq \frac{r(s^2 + 4Rr + r^2)}{s^2 - 8Rr + r^2} \geq \min\{r_a, r_b, r_c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = s - a, y = s - b, z = s - c \text{ and then : } a = y + z, b = z + x,$$
$$c = x + y \;\&\; s = x + y + z; \text{ furthermore, for proving } \max\{r_a, r_b, r_c\} \overset{?}{\geq} \sqrt{4R^2 - 7r^2},$$
$$\text{we assume WLOG } r_a = \max\{r_a, r_b, r_c\} \therefore r_a \geq r_b, r_c \Rightarrow a \geq b, c$$

$$\Rightarrow y+z\ge z+x, x+y \Rightarrow \boxed{y,z\ge x}\ \text{and hence, we can assign :}$$

$$y=x+m\ \text{and}\ z=x+n\ (m,n\ge 0)\ \text{and then : } max\{r_a,r_b,r_c\}\overset{?}{\ge}\sqrt{4R^2-7r^2}$$

$$\Leftrightarrow \frac{s^2}{(s-a)^2}\overset{?}{\ge}\frac{4R^2}{r^2}-7 \Leftrightarrow \frac{(x+y+z)^2}{x^2}\overset{?}{\ge}\frac{(y+z)^2(z+x)^2(x+y)^2}{4x^2y^2z^2}-7$$

$$\Leftrightarrow 4(x+m)^2(x+n)^2(3x+m+n)^2+28x^2(x+m)^2(x+n)^2-$$
$$(3x+m+n)^2(2x+m)^2 2(x+n)^2\overset{?}{\ge}0$$

$$\Leftrightarrow 3m^4n^2+4m^4nx+6m^3n^3+32m^3n^2x+32m^3nx^2+3m^2n^4+32m^2n^3x+$$
$$124m^2n^2x+128m^2nx^3+20m^2x^4+4mn^4x+32mn^3x^2+128mn^2x^3+$$
$$136mnx^4+24mx^5+20n^2x^4+24nx^5\overset{?}{\ge}0\to \text{true} \because m,n\ge 0$$

$$\therefore \boxed{max\{r_a,r_b,r_c\}\ge\sqrt{4R^2-7r^2}}\ \text{and now, for proving}\ \frac{r(s^2+4Rr+r^2)}{s^2-8Rr+r^2}\overset{?}{\ge}\min\{r_a,r_b,r_c\},$$

$$\text{we assume WLOG } r_a=\min\{r_a,r_b,r_c\}\therefore r_a\le r_b,r_c\Rightarrow a\le b,c$$

$$\Rightarrow y+z\le z+x,x+y\Rightarrow\boxed{y,z\le x}\ \text{and then : }\frac{r(s^2+4Rr+r^2)}{s^2-8Rr+r^2}\overset{?}{\ge}\min\{r_a,r_b,r_c\}$$

$$\Leftrightarrow\frac{r\left(\left(\sum_{cyc}x\right)^2+(y+z)(z+x)(x+y)+xyz\right)}{\left(\sum_{cyc}x\right)^2-2(y+z)(z+x)(x+y)+xyz}\overset{?}{\ge}\frac{r(x+y+z)}{x}$$

$$\Leftrightarrow 2x^3(y+z)+2x^2(y^2+z^2+2yz)-x(y^3+z^3+y^2z+yz^2)-$$
$$y^4-z^4-2y^2z^2-2y^3z-2yz^3\overset{?}{\ge}0$$

$$\Leftrightarrow 2y(x^3-z^3)+2z(x^3-y^3)+y^2(x^2-y^2)+z^2(x^2-z^2)+$$
$$2yz(x^2-yz)+x^2(y+z)^2-x(y+z)(y^2+z^2)\overset{?}{\ge}0$$

$$\Leftrightarrow 2y(x^3-z^3)+2z(x^3-y^3)+y^2(x^2-y^2)+z^2(x^2-z^2)+2yz(x^2-yz)+$$
$$x(y+z)\big(y(x-y)+z(x-z)\big)\overset{?}{\ge}0\to\text{true}\because x\ge y,z$$

$$\therefore\boxed{\frac{r(s^2+4Rr+r^2)}{s^2-8Rr+r^2}\ge\min\{r_a,r_b,r_c\}}\ \text{and finally,}\ \frac{r(s^2+4Rr+r^2)}{s^2-8Rr+r^2}\overset{\text{Gerretsen}}{\le}$$

$$\frac{r(4R^2+8Rr+4r^2)}{8Rr-4r^2}\overset{?}{\le}\sqrt{4R^2-7r^2}\Leftrightarrow 15t^4-20t^3-30t^2+24t-8\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(15t^3+10t^2-10t+4)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2$$

$$\therefore\boxed{\sqrt{4R^2-7r^2}\ge\frac{r(s^2+4Rr+r^2)}{s^2-8Rr+r^2}}\ \text{and so,}$$

$$max\{r_a,r_b,r_c\}\ge\sqrt{4R^2-7r^2}\ge\frac{r(s^2+4Rr+r^2)}{s^2-8Rr+r^2}\ge\min\{r_a,r_b,r_c\}\ \forall\,\Delta\,ABC,$$
$$''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

4117. In any Δ ABC the following relationship holds :

$$\sqrt{g_a m_a} \le \sqrt{\frac{3R-r}{5r}} \cdot h_a$$

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Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$
$c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$
and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$,
$y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2)$,
$y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right), y^6 + z^6 = x^6((m^3 - 3nn)^2 - 2n^3)$,

$$\text{and now, } \sqrt{g_a m_a} \overset{?}{\le} \sqrt{\frac{3R-r}{5r}} . h_a \overset{\text{via Bogdan Fustei}}{\Leftrightarrow}$$

$$\left((s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}\right)\left(s(s-a) + \frac{(b-c)^2}{4}\right) \overset{?}{\le}$$
$$\left(\frac{3}{5} . \frac{(y+z)(z+x)(x+y)}{4xyz} - \frac{1}{5}\right)^2 . \frac{16(x+y+z)^2 x^2 y^2 z^2}{(y+z)^4}$$
$$\Leftrightarrow (x^2(y+z) + 4xyz)\left(\frac{4x(x+y+z) + (y-z)^2}{4}\right) \overset{?}{\le}$$
$$\frac{(3(y+z)(z+x)(x+y) - 4xyz)^2}{400} . \frac{16(x+y+z)^2}{(y+z)^3}$$
$$\Leftrightarrow 36x^6(y+z)^2 + 144x^5(y^3+z^3) + 336x^5.yz(y+z) + 116x^4(y^4+z^4) +$$
$$248x^4.yz(y^2+z^2) + 328x^4.y^2z^2 + 44x^3(y^5+z^5) - 252x^3.yz(y^3+z^3) -$$
$$944x^3.y^2z^2(y+z) + 11x^2(y^6+z^6) - 114x^2.yz(y^4+z^4) - 647x^2.y^2z^2(y^2+z^2)$$
$$-1044x^2.y^3z^3 - 8x.yz(y^5+z^5) + 236x.y^2z^2(y^3+z^3) + 848x.y^3z^3(y+z) +$$
$$36y^2z^2(y+z)^4 2 \boxed{\underset{(1)}{\overset{?}{\ge}}} 0 \text{ and via set of relations "S", in order to prove (1),}$$

$$\text{suffices to prove, via simplification : } \overbrace{(36m^4 + 376m^3 - 92m^2 + 32m + 64)}^{\sigma_1} n^2 -$$
$$\overbrace{(28m^5 + 180m^4 + 472m^3 + 216m^2 + 96m)}^{\sigma_2} n +$$
$$\overbrace{(11m^6 + 44m^5 + 116m^4 + 144m^3 + 36m^2)}^{\sigma_3} + \boxed{\underset{(2)}{\overset{?}{\ge}}} 0$$

Now, discriminant of : $376m^2 - 92m + 32 = (92)^2 - (128)(376) < 0$

$$\Rightarrow 376m^3-92m^2+32m>0\Rightarrow \sigma_1>0 \text{ and } \because \text{LHS of } ② \text{ is a quadratic}$$
$$\text{polynomial in "n" with discriminant, } \boxed{\delta}=\sigma_2^2-4\sigma_1\sigma_3=$$
$$-800m^4(m+1)^2(m^4+14m^3-4m^2-8m-128), \therefore ② \text{ is trivially true if :}$$
$$m^4+14m^3-4m^2-8m-128\geq 0 \text{ (making } \delta\leq 0) \text{ and when :}$$
$$\boxed{m^4+14m^3-4m^2-8m-128\overset{(*)}{<}0}\ (\delta>0), \text{then, in order to prove } ②,$$
$$\text{it suffices to prove : } 2\sigma_1.n\overset{?}{\leq}\sigma_2-\sqrt{\delta}\ \&\text{ since } n\overset{AM-GM}{\leq}\frac{m^2}{4}\ \therefore \text{ it suffices to prove :}$$
$$\sigma_1.\frac{m^2}{2}\overset{?}{\leq}\sigma_2-\sqrt{\delta}\Leftrightarrow -2m(m+1)(9m^4+71m^3-184m^2-44m-48)\ \boxed{\overset{?}{\geq} ③}$$
$$\sqrt{-800m^4(m+1)^2(m^4+14m^3-4m^2-8m-128)}$$
$$\text{Now, } m^4+14m^3-4m^2-8m-128<0 \text{ (we are considering)}\Rightarrow$$
$$\frac{1}{4}\left(m^3+\frac{65m^2}{4}+\frac{521m}{16}+\frac{4177}{64}\right)(4m-9)+\frac{4825}{256}<0$$
$$\Rightarrow\frac{1}{4}\left(m^3+\frac{65m^2}{4}+\frac{521m}{16}+\frac{4177}{64}\right)(4m-9)<-\frac{4825}{256}<0\Rightarrow 4m-9<0\rightarrow(\bullet)$$
$$\text{We have : } 9m^4+71m^3-184m^2-44m-48=$$
$$\left(\frac{4m-9}{4}\right)\left(9m^3+\frac{365m^2}{4}+\frac{341m}{16}+\frac{253}{64}\right)-\frac{10011}{64}\overset{\text{via }(*)}{<}-\frac{10011}{64}<0$$
$$\Rightarrow\boxed{-2m(m+1)(9m^4+71m^3-184m^2-44m-48)\overset{(**)}{>}0}$$
$$\therefore (*) \text{ and } (**)\Rightarrow \text{in order to prove } ③, \text{it suffices to prove :}$$
$$4m^2(m+1)^2(9m^4+71m^3-184m^2-44m-48)^2\overset{?}{\geq}$$
$$-800m^4(m+1)^2(m^4+14m^3-4m^2-8m-128)$$
$$\Leftrightarrow\boxed{12m^2(m+1)^2(m-2)^2(9m+4)(3m^5+58m^4+271m^3-64m^2+28m+48)\overset{?}{\geq}0}$$
$$\rightarrow \text{true} \because \text{discriminant of : } 271m^2-64m+28=(64)^2-(112)(271)<0$$
$$\Rightarrow 271m^3-64m^2+28m>0\Rightarrow 3m^5+58m^4+271m^3-64m^2+28m+48>0$$
$$\Rightarrow ③\Rightarrow ②\Rightarrow ① \text{ is true }\therefore\sqrt{g_a m_a}\leq\sqrt{\frac{3R-r}{5r}}.h_a\ \forall\,\Delta ABC,$$
$$\text{"=" iff } b=c \text{ and } b+c=2a\Rightarrow\text{"=" iff }\Delta ABC \text{ is equilateral (QED)}$$

4118. In any Δ ABC the following relationship holds :

$$\sqrt{w_a m_a}\leq\sqrt{\frac{R}{r}-1}\cdot h_a$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=s-a, y=s-b, z=s-c \text{ and then : } a=y+z, b=z+x,$$
$$c=x+y \text{ and } s=x+y+z \text{ and furthermore, we denote : } \frac{y+z}{x}=m \text{ and } \frac{yz}{x^2}=n$$
$$\text{and then, we have the following set "S" of relations :}$$
$$y^2+z^2=x^2(m^2-2n), y^3+z^3=x^3(m^3-3nn),$$
$$y^4+z^4=x^4((m^2-2n)^2-2n^2), y^5+z^5=x^5\left(m((m^2-2n)^2+n^2-nm^2)\right),$$
$$y^6+z^6=x^6((m^3-3nn)^2-2n^3), \text{ and now, } w_a m_a \le w_a \frac{\sqrt{s^2-11r^2}}{4r}.h_a$$
$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Dang Ngoc Minh}-77;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right) \overset{?}{\le} \left(\frac{R}{r}-1\right)h_a^2$$
$$\Leftrightarrow \frac{1}{\cos^2\frac{B-C}{2}} \overset{?}{\le} \frac{16(s-a)(s-b)(s-c)\left((y+z)(z+x)(x+y)-4xyz\right)^2}{\left(s^3-11(s-a)(s-b)(s-c)\right).16x^2y^2z^2}$$
$$\Leftrightarrow \frac{(y+z)^2(z+x)(x+y)}{(2x+y+z)^2yz} \overset{?}{\le} \frac{\left((y+z)(z+x)(x+y)-4xyz\right)^2}{((x+y+z)^3-11xyz).xyz}$$
$$\Leftrightarrow 3x^6(y+z)^2+8x^5(y^3+z^3)-8x^5.yz(y+z)+7x^4(y^4+z^4)+14x^4.y^2z^2-$$
$$18x^4.yz(y^2+z^2)+2x^3\left(y^5+z^5\right)-6x^3.yz(y^3+z^3)+4x^3.y^2z^2(y+z)-$$
$$x^2.yz(y^4+z^4)-5x^2.y^2z^2(y^2+z^2)-8x^2.y^3z^3+x.yz\left(y^5+z^5\right)+$$
$$x.y^2z^2(y^3+z^3)-2x.y^3z^3(y+z)+y^2z^2(y+z)^4 \boxed{\underset{(1)}{\overset{?}{\ge}}} 0 \text{ and via}$$
$$\text{set of relations "S", to prove (1), it suffices to prove, following simplification :}$$
$$\overbrace{(m^4-4m^3-m^2+32m+64)}^{\Omega_1}n^2+\overbrace{(m^5-m^4-16m^3-46m^2-32m)}^{\Omega_2}n+$$
$$\overbrace{(2m^5+7m^4+8m^3+3m^2)}^{\Omega_3} \boxed{\underset{(2)}{\overset{?}{\ge}}} 0$$
$$\text{Now, } \Omega_1=\frac{(m^2-4m)^2+(m^2-9)^2+64m+47}{2} \overset{m>0}{>} 0 \text{ and furthermore,}$$
$$\text{we observe that, if : } \Omega_2 \ge 0, \text{ then (2) is trivially true and so, we now focus on :}$$
$$\Omega_2<0 \text{ and then : } \Omega_2=(m^4+5m^3+14m^2+38m+196)(m-6)+1176<0$$
$$\Rightarrow (m^4+5m^3+14m^2+38m+196)(m-6)<-1176<0 \Rightarrow \boxed{m<6} \to (*);$$
$$\text{also LHS of (2) is a quadratic polynomial in "n" with discriminant, } \delta=$$
$$\Omega_2^2-4\Omega_1.\Omega_3=m^2(m+1)^2\left(m^6-12m^5-4m^4+48m^3+80m^2+256\right)$$
$$\text{and so, when : } m^6-12m^5-4m^4+48m^3+80m^2+256 \le 0, \text{ then : } \delta \le 0$$
$$\text{and (2) is trivially true and so, we now focus on : } P=$$
$$m^6-12m^5-4m^4+48m^3+80m^2+256>0\ (\delta>0)$$
$$\Rightarrow \left(m^5-9m^4-31m^3-45m^2-55m-165\right)(m-3)>239>0 \to (\bullet)$$
$$m^5-9m^4-31m^3-45m^2-55m-165=$$

$$m^4(m-6-3)-31m^3-45m^2-55m-165\overset{via\ (*)}{<}0\overset{via\ (\bullet)}{\Rightarrow}\boxed{m<3}\to(**)$$

Now, in order to prove ②, it suffices to prove : $2\Omega_1.n\overset{?}{\leq}-\Omega_2-\sqrt{\delta}$ *and* $\because$

$$n\overset{AM-GM}{\leq}\frac{m^2}{4}\ \therefore \text{ it suffices to prove : } m^2\Omega_1\overset{?}{\leq}-2\Omega_2-2\sqrt{\delta}$$

$$\Leftrightarrow 2\sqrt{\delta}\boxed{\underset{③}{\overset{?}{\leq}}}m(-m^5+2m^4+3m^3+28m+64)\ \textit{and now,}$$

$$-m^5+2m^4+3m^3+28m+64=$$

$$\frac{m^4(8-m)}{4}+\frac{m^3(12-m^2)}{4}+\frac{m(112-m^4)}{4}+\frac{256-m^5}{4}\overset{m<3,via\ (**)}{>}0$$

$\therefore$ in order to prove ③, it suffices to prove :

$$m^2(-m^5+2m^4+3m^3+28m+64)^2\overset{?}{\geq}4m^2(m+1)^2\begin{pmatrix}m^6-12m^5-4m^4+\\48m^3+80m^2+256\end{pmatrix}\Leftrightarrow$$

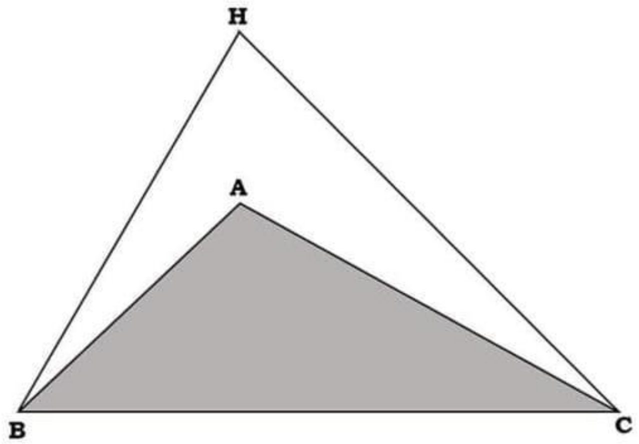
$$\boxed{m^2(m+2)^2(m-2)^2(2m^3(10-m^2)+2m^2(30-m^3)+m^6+2m^4+m^2+96m+192)\overset{?}{\geq}0}$$

$$\to \text{true} \because m<3, via\ (**) \Rightarrow ③\Rightarrow ②\Rightarrow ① \text{ is true} \therefore \sqrt{w_a m_a}\leq\sqrt{\frac{R}{r}-1}.h_a$$

$\forall\ \Delta\ ABC$, " = " iff $\Delta\ ABC$ is equil*ateral* (QED)

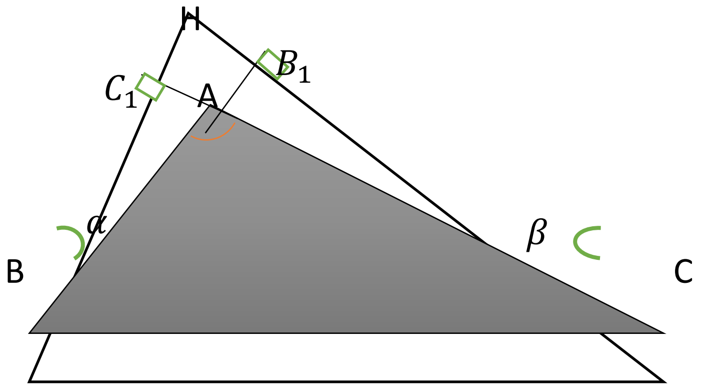
4119. In the figure below ,the point H is the orthocenter of the triangle ABC ,whose sides fulfill the condition :

$$\frac{AC}{AB}=\frac{BC}{AC}=\sqrt{2}\ \text{.That said , show that } \frac{[HBAC]}{[ABC]}=\frac{8}{7}$$



Proposed by Rachid Iksi-Morocco

Solution by Mirsadix Muzefferov-Azerbaijan



$$In\ \Delta ABC\ law\ cosine:\ \cos\theta=\frac{AB^2+AC^2-BC^2}{2AB\cdot AC}=\frac{\frac{BC^2}{4}+\frac{BC^2}{2}-BC^2}{2\cdot\frac{BC}{\sqrt{2}}\cdot\frac{BC}{2}}$$

$$AB=\frac{AC}{\sqrt{2}},AC=\frac{BC}{\sqrt{2}},AB=\frac{BC}{2}$$

$$\cos\theta=\frac{(1+2-4)\sqrt{2}}{4}=-\frac{\sqrt{2}}{4}$$

$$S(ABC)=\frac{1}{2}BC^2\cdot\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}=\frac{1}{2}BC^2\frac{\sin\alpha\sin\beta}{\sin\theta}$$

$$S(HBC)=\frac{1}{2}BC^2\frac{\sin(\frac{\pi}{2}-\beta)\sin(\frac{\pi}{2}-\alpha)}{\sin(\pi-(\alpha+\beta))}=\frac{1}{2}BC^2\frac{\cos\alpha\cdot\cos\beta}{\sin\theta}$$

$$\frac{S(HBAC)}{S(ABC)}=\frac{\frac{1}{2}BC^2\frac{\cos\alpha\cdot\cos\beta}{\sin\theta}-\frac{1}{2}BC^2\frac{\sin\alpha\sin\beta}{\sin\theta}}{\frac{1}{2}BC^2\frac{\sin\alpha\sin\beta}{\sin\theta}}=\frac{\cos(\alpha+\beta)}{\sin\alpha\sin\beta}=-\frac{\cos\theta}{\sin\alpha\sin\beta}$$

$$In\ \Delta ABC\ law\ sine:\ \frac{BC}{\sin\theta}=\frac{AC}{\sin\alpha}\Rightarrow\frac{BC}{AC}=\frac{\sin\theta}{\sin\alpha}=\sqrt{2}\Rightarrow\sin\alpha=\frac{\sin\theta}{\sqrt{2}}$$

$$\frac{AC}{\sin\alpha}=\frac{AB}{\sin\beta}\Rightarrow\frac{AC}{AB}=\frac{\sin\alpha}{\sin\beta}=\sqrt{2}\Rightarrow\sin\beta=\frac{\sin\alpha}{\sqrt{2}}=\frac{\sin\theta}{2}$$

$$\frac{S(HBAC)}{S(ABC)}=-\frac{\cos\theta}{\frac{\sin\theta}{\sqrt{2}}\cdot\frac{\sin\theta}{2}}=-2\sqrt{2}\cdot\frac{\cos\theta}{sin^2\theta}=-2\sqrt{2}\cdot\frac{-\frac{\sqrt{2}}{4}}{\frac{7}{8}}=\frac{8}{7}$$

$$sin^2\theta=1-cos^2\theta=1-\frac{2}{16}=\frac{7}{8}$$

4120. In any Δ ABC the following relationship holds :

$$7\ge\sum_{cyc}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{p_a}{l_a}}}}}\right)$$

$$+\sum_{cyc}\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality}\Rightarrow g_a\le AI+r\overset{?}{\le}l_a\Leftrightarrow\frac{r}{\sin\frac{A}{2}}+r\overset{?}{\le}\frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow\frac{r}{\sin\frac{A}{2}}+r\overset{?}{\le}\frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}}\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+1\overset{?}{\le}\frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+1\overset{?}{\le}\frac{a}{(b+c)\sin\frac{A}{2}}+\frac{1}{\sin\frac{A}{2}}\Leftrightarrow(b+c)\sin\frac{A}{2}\overset{?}{\le}a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}\overset{?}{\le}4R\sin\frac{A}{2}\cos\frac{A}{2}\Leftrightarrow\cos\frac{B-C}{2}\overset{?}{\le}1\to\text{true}\therefore l_a\ge g_a\to(1)$$

$$\text{Again, }\sum_{cyc}\frac{AI}{l_a}=\sum_{cyc}\frac{r(b+c)}{\sin\frac{A}{2}.2bc\cos\frac{A}{2}}=\sum_{cyc}\frac{2R.r(b+c)}{abc}$$

$$=\frac{2Rr}{4Rrs}.\sum_{cyc}(b+c)=\frac{8Rrs}{4Rrs}=2\therefore\sum_{cyc}\frac{AI}{l_a}=2\to(2)$$

$$\text{Now, }\sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a}\overset{\text{Bogdan Fustei}}{\ge}\sum_{cyc}\frac{n_a-AI+r-g_a}{l_a}$$

(Reference : Inequality in Triangle by Bogdan Fustei – 103; published at www.ssmrmh.ro)

$$\overset{\text{Triangle inequality}}{\ge}\sum_{cyc}\frac{n_a-AI+r-AI-r}{l_a}=\sum_{cyc}\frac{n_a}{l_a}-\sum_{cyc}\frac{2AI}{l_a}\overset{\text{via }(2)}{=}\sum_{cyc}\frac{n_a}{l_a}-4$$

$$\Rightarrow 7+\sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a}\ge 3+\sum_{cyc}\frac{n_a}{l_a}=\sum_{cyc}\frac{n_a+l_a}{l_a}$$

$$\overset{\text{via ① and analogs}}{\geq}\sum_{cyc}\frac{n_a+g_a}{l_a}\overset{\text{Bogdan Fustei (2019)}}{\geq}\sum_{cyc}\frac{2m_a}{l_a}\to(*)$$

$$Also,\frac{2m_a}{l_a}\overset{\text{Lascu}}{\geq}\frac{(b+c)\cos\frac{A}{2}}{\frac{2bc\cos\frac{A}{2}}{b+c}}=\frac{(b+c)^2}{2bc}\Rightarrow\frac{m_a}{l_a}\boxed{\overset{(**)}{\geq}}\left(\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\right)^2=\sigma^2\ (say)$$

$$p_a^2l_a^2\overset{\substack{\text{Fustei}\\\text{and}\\\text{Ben Ajiba}}}{=}\left(s(s-a)+\frac{s(3s+a)}{(2s+a)^2}(b-c)^2\right)\left(s(s-a)-\frac{s(s-a)}{(2s-a)^2}.(b-c)^2\right)$$

$$\overset{?}{\leq}m_a^4=s^2(s-a)^2+\frac{(b-c)^4}{16}+\frac{s(s-a)(b-c)^2}{2}\Leftrightarrow$$

$$\left(\frac{1}{16}+\frac{s^2(3s+a)(s-a)}{(4s^2-a^2)^2}\right).(b-c)^2+\frac{s(s-a)a\left((s-a)(16s^2+4sa)+a^3\right)}{2(4s^2-a^2)^2}$$

$$\overset{?}{\geq}0\to\text{true}\because s>a\Rightarrow p_al_a\leq m_a^2\Rightarrow\sqrt[4]{\frac{p_a}{l_a}}\leq\sqrt{\frac{m_a}{l_a}}\therefore\text{in order to prove}:\frac{2m_a}{l_a}\overset{?}{\underset{(***)}{\geq}}$$

$$\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{p_a}{l_a}}}}},\text{suffices}$$

$$\frac{2m_a}{l_a}\overset{?}{\geq}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt{\frac{m_a}{l_a}}}}}$$

$$\Leftrightarrow\left(\frac{m_a}{l_a}\right)^{\frac{127}{128}}\overset{?}{\geq}\sigma^{\frac{85}{64}}\text{ and via }(**),\text{it suffices to prove}:\sigma^{\frac{127}{64}}\overset{?}{\geq}\sigma^{\frac{85}{64}}\Leftrightarrow\sigma^{\frac{42}{64}}\overset{?}{\geq}1\to\text{true}$$

$$\because\sigma=\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\overset{\text{AM-GM}}{\geq}1\overset{\text{summation }\&\ (*),(***)}{\Rightarrow}7+\sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a}\geq$$

$$\sum_{cyc}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right).\sqrt[4]{\frac{p_a}{l_a}}}}}\right)$$

$\forall\Delta ABC,''='' \text{ iff }\Delta ABC$ is equilateral (QED)

4121. **In any ΔABC prove that the following relationship holds:**

$$\tan\frac{A}{4}+\tan\frac{B}{4}+\tan\frac{C}{4}\geq\frac{n_a+n_b+n_c-s-\sum_{cyc}\sqrt{(b-c)^2+r^2}}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Jenish Rijal-Nepal

Let I be the incenter, D be the projection of I on BC, and T_a be the A-excircle tangency point on BC. Then: $ID=r$, $BD=s-b$, $BT_a=s-c$, and $AT_a=n_a$.

Since $DT_a=|(s-c)-(s-b)|=|b-c|$, Pythagoras on $\triangle IDT_a$ yields:

$$IT_a=\sqrt{r^2+(b-c)^2}$$

By the triangle inequality on $\triangle AIT_a$, we have:

$$n_a\leq AI+IT_a\Rightarrow n_a-\sqrt{(b-c)^2+r^2}\leq AI$$

Summing cyclically for all three vertices, we have:

$$\sum_{cyc}n_a-\sum_{cyc}\sqrt{(b-c)^2+r^2}\overbrace{\leq}^{(1)}\sum_{cyc}AI$$

Via $AI=\dfrac{r}{\sin\frac{A}{2}}$ & $s=r\sum_{cyc}\cot\frac{A}{2}$ and we also subtract s from both sides of (1) & divide by r:

$$\frac{\sum_{cyc}n_a-s-\sum_{cyc}\sqrt{(b-c)^2+r^2}}{r}\leq\sum_{cyc}\left(\frac{1}{\sin\frac{A}{2}}-\cot\frac{A}{2}\right)$$

Via half $-$ angle identity $\dfrac{1}{\sin\frac{A}{2}}-\cot\dfrac{A}{2}=\tan\dfrac{A}{4}$, we have:

$$\frac{n_a+n_b+n_c-s-\sum_{cyc}\sqrt{(b-c)^2+r^2}}{r}\leq\tan\frac{A}{4}+\tan\frac{B}{4}+\tan\frac{C}{4}$$

Equality holds if and only if the triangle is equilateral.

4122. In any Δ ABC the following relationship holds :

$$7 \ge \sum_{cyc}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\right)+\sum_{cyc}\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality} \Rightarrow g_a \le AI + r \overset{?}{\le} l_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}}+r \overset{?}{\le} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}}+r \overset{?}{\le} \frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}}+1 \overset{?}{\le} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}}+1 \overset{?}{\le} \frac{a}{(b+c)\sin\frac{A}{2}}+\frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\le} a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\le} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \overset{?}{\le} 1 \rightarrow \text{true} \therefore l_a \ge g_a \rightarrow (1)$$

$$\text{Again}, \sum_{cyc}\frac{AI}{l_a}=\sum_{cyc}\frac{r(b+c)}{\sin\frac{A}{2}\cdot 2bc\cos\frac{A}{2}}=\sum_{cyc}\frac{2R.r(b+c)}{abc}$$

$$=\frac{2Rr}{4Rrs}\cdot\sum_{cyc}(b+c)=\frac{8Rrs}{4Rrs}=2 \therefore \sum_{cyc}\frac{AI}{l_a}=2 \rightarrow (2)$$

$$\text{Now}, \sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \overset{\text{Bogdan Fustei}}{\ge} \sum_{cyc}\frac{n_a-AI+r-g_a}{l_a}$$

(Reference : Inequality in Triangle by Bogdan Fustei – 103; published at www.ssmrmh.ro)

$$\overset{\text{Triangle inequality}}{\ge} \sum_{cyc}\frac{n_a-AI+r-AI-r}{l_a}=\sum_{cyc}\frac{n_a}{l_a}-\sum_{cyc}\frac{2AI}{l_a} \overset{\text{via (2)}}{=} \sum_{cyc}\frac{n_a}{l_a}-4$$

$$\Rightarrow 7+\sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a} \ge 3+\sum_{cyc}\frac{n_a}{l_a}=\sum_{cyc}\frac{n_a+l_a}{l_a}$$

$$\overset{\text{via (1) and analogs}}{\ge} \sum_{cyc}\frac{n_a+g_a}{l_a} \overset{\text{Bogdan Fustei (2019)}}{\ge} \sum_{cyc}\frac{2m_a}{l_a} \overset{\text{Lascu}}{\ge} \sum_{cyc}\frac{(b+c)\cos\frac{A}{2}}{\frac{2bc\cos\frac{A}{2}}{b+c}}$$

$$=\sum_{cyc}\frac{(b+c)^2}{2bc}\ \therefore\ 7+\sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a}\boxed{\overset{(*)}{\geq}}\sum_{cyc}\frac{(b+c)^2}{2bc}$$

We shall now prove that : $\frac{(b+c)^2}{2bc}\overset{?}{\geq}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}$

$$\Leftrightarrow\frac{\left(x^2+\frac{1}{x^2}+2\right)}{2}\overset{?}{\geq}\left(x+\frac{1}{x}\right)\cdot\sqrt[4]{\frac{1}{2}\left(x+\frac{1}{x}\right)}\ \left(x=\sqrt{\frac{b}{c}}\right)$$

$$\Leftrightarrow\frac{(x^2+1)^8}{16x^8}\overset{?}{\geq}\frac{(x^2+1)^4}{x^4}\cdot\frac{x^2+1}{2x}\Leftrightarrow(x^2+1)^3\overset{?}{\geq}8x^3\Leftrightarrow x^2+1\overset{?}{\geq}2x$$

$$\Leftrightarrow(x-1)^2\overset{?}{\geq}0\to\text{true}\ \therefore\ \frac{(b+c)^2}{2bc}\overset{?}{\geq}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\ \text{and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{(b+c)^2}{2bc}\boxed{\overset{(**)}{\geq}}\sum_{cyc}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\right)\ \text{and so, }(*)\text{ and }(**)\Rightarrow$$

$$7+\sum_{cyc}\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a}\geq\sum_{cyc}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\right)$$

$$\Rightarrow 7\geq\sum_{cyc}\left(\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\cdot\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\right)+\sum_{cyc}\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$$

$\forall\ \Delta\ ABC$, " = " iff $\Delta\ ABC$ is equilateral (QED)

4123. In any $\Delta\ ABC$ the following relationship holds :

$$\sqrt[3]{g_a w_a n_a}\leq\sqrt{p_a m_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(p_a m_a)^3\geq\left(s(s-a)+\frac{1}{3}(b-c)^2\right)^3$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Dang Ngoc Minh} - 324;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right)$$

$$\therefore p_a^3m_a^3 \geq s^3(s-a)^3 + \frac{(b-c)^6}{27} + s^2(s-a)^2(b-c)^2 + \frac{s(s-a)(b-c)^4}{3} \rightarrow (1)$$

$$\text{Again, } n_a^2g_a^2w_a^2 \overset{\text{Bogdan Fustei}}{=}$$

$$\left(s(s-a)+\frac{s}{a}(b-c)^2\right)\left(s(s-a)-\frac{s-a}{a}.(b-c)^2\right)\left(\begin{matrix}s(s-a)-\\ \frac{s(s-a)}{(2s-a)^2}.(b-c)^2\end{matrix}\right)$$

$$=\left(s^2(s-a)^2+s(s-a)(b-c)^2-\frac{s(s-a)}{a^2}.(b-c)^4\right)\left(\begin{matrix}s(s-a)-\\ \frac{s(s-a)}{(2s-a)^2}.(b-c)^2\end{matrix}\right)$$

$$= s^3(s-a)^3+s^2(s-a)^2\left(\begin{matrix}\frac{-s(s-a)}{(2s-a)^2}.(b-c)^2+(b-c)^2-\frac{(b-c)^4}{(2s-a)^2}-\\ \frac{(b-c)^4}{a^2}+\frac{(b-c)^6}{a^2(2s-a)^2}\end{matrix}\right) \overset{?}{\leq}$$

$$s^3(s-a)^3+\frac{(b-c)^6}{27}+s^2(s-a)^2(b-c)^2+\frac{s(s-a)(b-c)^4}{3} \Leftrightarrow$$

$$(b-c)^6\left(\frac{s^2(s-a)^2}{a^2(2s-a)^2}-\frac{1}{27}\right)-(b-c)^4\left(\frac{s^2(s-a)^2}{(2s-a)^2}+\frac{s^2(s-a)^2}{a^2}+\frac{s(s-a)}{3}\right)-$$
$$\left(\frac{s^3(s-a)^3}{(2s-a)^2}-s^2(s-a)^2+s^2(s-a)^2\right)(b-c)^2 \overset{?}{\leq} 0$$

$$\Leftrightarrow \overbrace{(27s^4-54s^3a+23s^2a^2+4sa^3-a^4)}^{\sigma_1}(b-c)^4-$$
$$\overbrace{9s(s-a)(12s^4-24s^3a+22s^2a^2-10sa^3+a^4)}^{\sigma_2}(b-c)^2-\overbrace{27a^2s^3(s-a)^3}^{\sigma_3}\boxed{\overset{?}{\underset{(*)}{\leq}}}0$$

$(\because (b-c)^2 \geq 0)$ and it's trivially true if : $\sigma_1 \leq 0$

$$\left(\begin{matrix}\because 12s^4-24s^3a+22s^2a^2-10sa^3+a^4=\\ (s-a)(12s^2(s-a)+10sa^2)+a^4>0 \text{ as } (s-a)>0\end{matrix}\right) \text{ and when : } \sigma_1>0,$$

$$\text{then, since } \delta=\sigma_2^2+4\sigma_1\sigma_3=$$
$$(s-a)^2(2s-a)^2\left(\begin{matrix}(s-a)^2(108s^3(s-a)+180s^2a^2+44a^4)+\\ 44a^5(s-a)+3a^6\end{matrix}\right)>0$$

$\therefore$ in order to prove $(*)$, it suffices to prove :

$$2\sigma_1.(b-c)^2\boxed{\overset{?}{\underset{(i)}{\leq}}}\sigma_2+\sqrt{\delta} \text{ AND } 2\sigma_1.(b-c)^2\boxed{\overset{?}{\underset{(ii)}{\geq}}}\sigma_2-\sqrt{\delta}$$

$$\text{Now, } 2\sigma_1.(b-c)^2-\sigma_2<2\sigma_1.a^2-\sigma_2\leq|2\sigma_1.a^2-\sigma_2|\overset{?}{\leq}\sqrt{\delta}$$

$$\Leftrightarrow \delta \overset{?}{\geq} (2\sigma_1.a^2-\sigma_2)^2$$

$$\Leftrightarrow 3645t^{10}-18225t^9+37611t^8-41094t^7+25155t^6-8181t^5+$$

$$965t^4+212t^3-105t^2+17t-1\overset{?}{\geq}0\ \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow\begin{pmatrix}27t^4-54t^3+\\23t^2+4t-1\end{pmatrix}\left((t-1)\left((t-1)\begin{pmatrix}135t^3(t-1)+\\63t^2+13\end{pmatrix}+13\right)+1\right)\overset{?}{\geq}0\to\text{true}$$

$$\because t>1\text{ and }\sigma_1>0\Rightarrow 2\sigma_1.(b-c)^2-\sigma_2<\sqrt{\delta}\Rightarrow\text{(i) is true (strict inequality)}$$

$$\text{and finally, }2\sigma_1.(b-c)^2+\sqrt{\delta}\geq\sqrt{\delta}=\sqrt{\sigma_2^2+4\sigma_1\sigma_3}>\sqrt{\sigma_2^2}=\sigma_2$$

$$(\because\sigma_2>0\text{ as }12s^4-24s^3a+22s^2a^2-10sa^3+a^4>0;\text{ demonstrated earlier})$$

$$\Rightarrow\text{(ii) is true (strict inequality)}\therefore\text{both (i) and (ii) are true}\Rightarrow(*)\text{ is true}$$

$$\therefore n_a^2g_a^2w_a^2\leq s^3(s-a)^3+\frac{(b-c)^6}{27}+s^2(s-a)^2(b-c)^2+\frac{s(s-a)(b-c)^4}{3}\overset{\text{via }①}{\leq}$$

$$p_a^3m_a^3\Rightarrow\sqrt[3]{g_aw_an_a}\leq\sqrt{p_am_a}\ \forall\ \Delta ABC,''=''\text{ iff }b=c\text{ (QED)}$$

4124. In any Δ ABC the following relationship holds :

$$m_ag_a^2\geq h_ar_br_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2g_a^4\overset{?}{\geq}h_a^2.s^2(s-a)^2\overset{\text{Bogdan Fustei}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)-\frac{(s-a)(b-c)^2}{a}\right)^2\overset{?}{\geq}$$

$$\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)s^2(s-a)^2$$

$$\Leftrightarrow\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s^2(s-a)^2+\frac{(s-a)^2(b-c)^4}{a^2}-\frac{2s(s-a)^2(b-c)^2}{a}\right)\overset{?}{\geq}$$

$$s^3(s-a)^3-\frac{s^3(s-a)^3}{a^2}.(b-c)^2$$

$$\Leftrightarrow\frac{s(s-a)(b-c)^2}{a^2}-\frac{2s^2(s-a)}{a}+\frac{s^2}{4}+\frac{(b-c)^4}{4a^2}-\frac{s(b-c)^2}{2a}+\frac{s^3(s-a)}{a^2}\overset{?}{\geq}0$$

$$(\because(b-c)^2\geq 0)$$

$$\Leftrightarrow\frac{(b-c)^4}{4a^2}+\frac{2s^2-3sa}{2a^2}.(b-c)^2+\frac{s^2}{4a^2}.(4s(s-a)-8a(s-a)+a^2)\overset{?}{\geq}0$$

$$\Leftrightarrow(b-c)^4+2(b-c)^2.s(2s-3a)+s^2(2s-3a)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\left((b-c)^2+s(2s-3a)\right)^2\overset{?}{\geq}0\to\text{true}\therefore m_ag_a^2\geq h_ar_br_c\ \forall\ \Delta ABC,$$

4125. In any Δ ABC the following relationship holds :

$$\min\{m_a,m_b,m_c\} \le \frac{\sqrt{s^2-4Rr+17r^2}}{2} \le \sqrt{s^2-7Rr-4r^2} \le \max\{m_a,m_b,m_c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x$, $c = x+y$ and $s = x+y+z$ and furthermore, for proving $\max\{m_a,m_b,m_c\} \overset{?}{\ge} \sqrt{s^2-7Rr-4r^2}$, we assume WLOG $m_a = \max\{m_a,m_b,m_c\}$ $\therefore m_a \ge m_b, m_c \Rightarrow a \le b,c \Rightarrow y+z \le z+x, x+y \Rightarrow \boxed{x \ge y,z}$

$$\text{Now, } m_a^2 \overset{?}{\ge} s^2-7Rr-4r^2 \Leftrightarrow$$

$$\frac{4x(x+y+z)+(y-z)^2}{4} \overset{?}{\ge} \left(\sum_{cyc} x\right)^2 - \frac{7(y+z)(z+x)(x+y)}{4(x+y+z)} - \frac{4xyz}{x+y+z}$$

$$\Leftrightarrow x^2y + x^2z + 4xyz \overset{?}{\ge} y^3 + z^3 + 2y^2z + 2yz^2$$

$$\Leftrightarrow y(x^2-y^2) + z(x^2-z^2) + 2yz\big((x-y)+(x-z)\big) \overset{?}{\ge} 0 \rightarrow \text{true} \because x \ge y,z$$

$\therefore \boxed{\max\{m_a,m_b,m_c\} \ge \sqrt{s^2-7Rr-4r^2}}$ and now, for proving $\min\{m_a,m_b,m_c\} \overset{?}{\le} \frac{\sqrt{s^2-4Rr+17r^2}}{2}$, we assume WLOG $m_a = \min\{m_a,m_b,m_c\} \therefore m_a \le m_b, m_c$ $\Rightarrow a \ge b,c \Rightarrow y+z \ge z+x, x+y \Rightarrow \boxed{y,z \ge x}$ and then, $m_a^2 \overset{?}{\le} \frac{s^2-4Rr+17r^2}{4}$

$$\Leftrightarrow \frac{4x(x+y+z)+(y-z)^2}{4} \overset{?}{\le} \frac{(x+y+z)^3-(y+z)(z+x)(x+y)+17xyz}{4(x+y+z)}$$

$$\Leftrightarrow y^2z + yz^2 + 5xyz \overset{?}{\ge} x^3 + 2x^2y + 2x^2z + xy^2 + xz^2$$

$$\Leftrightarrow y^2(z-x) + z^2(y-x) + x(yz-x^2) + 2x\big(y(z-x)+z(y-x)\big) \overset{?}{\ge} 0 \rightarrow \text{true}$$

$\because y,z \ge x \therefore \boxed{\min\{m_a,m_b,m_c\} \le \frac{\sqrt{s^2-4Rr+17r^2}}{2}}$ and finally,

$$\frac{\sqrt{s^2-4Rr+17r^2}}{2} \overset{?}{\le} \sqrt{s^2-7Rr-4r^2} \Leftrightarrow s^2 \overset{?}{\ge} 8Rr+11r^2$$

$\Leftrightarrow (s^2-16Rr+5r^2) + 8r(R-2r) \overset{?}{\ge} 0 \rightarrow \text{true} \because s^2 \overset{\text{Gerretsen}}{\ge} 16Rr-5r^2$ and $R \overset{\text{Euler}}{\ge} 2r \therefore \boxed{\frac{\sqrt{s^2-4Rr+17r^2}}{2} \le \sqrt{s^2-7Rr-4r^2}}$ and so,

$$\min\{m_a,m_b,m_c\}\le\frac{\sqrt{s^2-4Rr+17r^2}}{2}\le\sqrt{s^2-7Rr-4r^2}\le max\{m_a,m_b,m_c\}$$
$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC$ is equilateral (QED)

4126. In any Δ ABC the following relationship holds :

$$\min\{w_a,w_b,w_c\}\le 3r\le\sqrt{s^2-9Rr}\le max\{w_a,w_b,w_c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a, y=s-b, z=s-c$ and then : $a=y+z, b=z+x$, $c=x+y$ and $s=x+y+z$ and furthermore, for proving $max\{w_a,w_b,w_c\}\overset{?}{\ge}\sqrt{s^2-9Rr}$, we assume WLOG $w_a=max\{w_a,w_b,w_c\}$ $\therefore w_a\ge w_b,w_c\Rightarrow a\le b,c\Rightarrow y+z\le z+x,x+y\Rightarrow\boxed{x\ge y,z}$

$$\text{Now, } w_a^2\overset{?}{\ge}s^2-9Rr$$
$$\Leftrightarrow\frac{4(z+x)(x+y)}{(2x+y+z)^2}.x\left(\sum_{cyc}x\right)\overset{?}{\ge}\left(\sum_{cyc}x\right)^2-\frac{9(y+z)(z+x)(x+y)}{4(x+y+z)}$$
$$\Leftrightarrow 20x^4y+20x^4z+20x^3y^2+56x^3yz+20x^3z^2+23x^2y^2z+23x^2yz^2\ \boxed{\overset{?}{\ge}}\ 15x^2y^3+15x^2z^3+19xy^4+19xz^4+10xy^2z^2+24xy^3z+24xyz^3+4y^5+4z^5+11y^4z+11yz^4+13y^3z^2+13y^2z^3$$
$$\Leftrightarrow 15x^2y(x^2-y^2)+15x^2z(x^2-z^2)+4y(x^4-y^4)+4z(x^4-z^4)+19xy^2(x^2-y^2)+19xz^2(x^2-z^2)+23xy^2z(x-y)+23xyz^2(x-z)+(xy^2+xz^2)(x^2-yz)+8xyz(x^2-yz)+13yz(x^3-y^2z)+13yz(x^3-yz^2)+11yz(x^3-y^3)+11yz(x^3-z^3)+xy(x^3-yz^2)+xz(x^3-y^2z)\overset{?}{\ge}0$$

$\rightarrow$ true $\because x\ge y,z\ \therefore\ \boxed{\boxed{max\{w_a,w_b,w_c\}\ge\sqrt{s^2-9Rr}}}$ and now, for proving $\min\{w_a,w_b,w_c\}\overset{?}{\le}3r$, we assume WLOG $w_a=\min\{w_a,w_b,w_c\}\ \therefore w_a\le w_b,w_c$ $\Rightarrow a\ge b,c\Rightarrow y+z\ge z+x,x+y\Rightarrow\boxed{y,z\ge x}$ and hence, we can assign $y=x+m$ and $z=x+n$ $(m,n\ge 0)$ and then : $\min\{w_a,w_b,w_c\}\overset{?}{\le}3r\Leftrightarrow$

$$\frac{4(z+x)(x+y)}{(2x+y+z)^2}.x\left(\sum_{cyc}x\right)\overset{?}{\le}\frac{9xyz}{x+y+z}$$
$$\Leftrightarrow 9(x+m)(x+n)(4x+m+n)^2\overset{?}{\ge}4(3x+m+n)^2(2x+n)(2x+m)$$
$$\Leftrightarrow 5mn(m^2+n^2)+x(m^3+n^3)+10m^2n^2+51mnx(m+n)+17x^2(m^2+n^2)+142mnx^2+48x^3(m+n)\ \boxed{\overset{?}{\ge}}\ 0$$
$\rightarrow$ true $\because m,n\ge 0\ \therefore\ \boxed{\boxed{\min\{w_a,w_b,w_c\}\le 3r}}$ and

$$finally, s^2-9Rr \overset{Gerretsen}{\geq} 7Rr-5r^2 \overset{Euler}{\geq} 14r^2-5r^2=9r^2 \Rightarrow \boxed{\sqrt{s^2-9Rr}\geq 3r}$$

$$and\ so, \min\{w_a, w_b, w_c\} \leq 3r \leq \sqrt{s^2-9Rr} \leq max\{w_a, w_b, w_c\}\ \forall\ \Delta\ ABC,$$
$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

4127. In acute ΔABC the following relationship holds:

$$\frac{\sqrt{ctg(A)}}{a}+\frac{\sqrt{ctg(B)}}{b}+\frac{\sqrt{ctg(C)}}{c} \leq \frac{\sqrt[4]{3}R}{4r^2}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{\sqrt{ctg(A)}}{a} \overset{CSB}{\leq} \left(\sum_{cyc}\frac{1}{a^2}\right)^{\frac{1}{2}} \cdot \left(\sum_{cyc} ctg(A)\right)^{\frac{1}{2}} \overset{Steining}{\leq}$$

$$\frac{1}{2r}\cdot\sqrt{\frac{p^2-4Rr-r^2}{2pr}} \overset{Gerretsen}{\leq} \frac{1}{2r}\sqrt{\frac{4R^2+4Rr+3r^2-4Rr-r^2}{2pr}}=$$

$$\frac{1}{2r}\sqrt{\frac{2R^2+r^2}{pr}} \overset{Euler}{\leq} \frac{1}{2r}\cdot\frac{3R}{2}\cdot\sqrt{\frac{1}{pr}} \overset{Mitrinovici}{\leq} \frac{3R}{4r}\sqrt{\frac{1}{3\sqrt{3}r.r}}=\frac{\sqrt[4]{3}.R}{4r^2}$$

$$Equality\ holds\ for: \quad a=b=c$$

4128. In ΔABC the following relationship holds:

$$(r_a)^{b^2+c^2}.(r_b)^{a^2+c^2}.(r_c)^{b^2+a^2} \leq \left(\frac{3R}{2}\right)^{18R^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$(r_a)^{b^2+c^2}.(r_b)^{a^2+c^2}.(r_c)^{b^2+a^2} \overset{Weighted\ \ AM-GM}{\leq}$$

$$\leq \left(\frac{r_a(b^2+c^2)+r_b(a^2+c^2)+r_c(b^2+a^2)}{2(a^2+b^2+c^2)}\right)^{2\sum a^2} \leq$$

$$In\ \Delta ABC\ wlog: \ a\leq b\leq c, \quad r_a\leq r_b\leq r_c$$
$$\rightarrow \{b^2+c^2 \geq a^2+c^2 \geq b^2+a^2$$

$$\overset{Chebyshev}{\lesssim}\left(\frac{\frac{1}{3}(r_a+r_b+r_c).2(a^2+b^2+c^2)}{2(a^2+b^2+c^2)}\right)^{2\sum a^2}=$$

$$=\left(\frac{1}{3}(4R+r)\right)^{2\sum a^2}\overset{Euler}{\lesssim}\left(\frac{1}{3}.\frac{9R}{2}\right)^{2\sum a^2}\overset{Leibniz}{\lesssim}\left(\frac{3R}{2}\right)^{18R^2}$$

Equality holds if $a=b=c$.

4129. In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\sqrt{a\cos A}\le\frac{3\sqrt[4]{12R^2}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{a\cos A}\overset{CBS}{\le}\sqrt{3}.\sqrt{\sum_{cyc}(a\cos A)}=\sqrt{3R}.\sqrt{\sum_{cyc}\sin 2A}=\sqrt{3R}.\sqrt{4.\frac{4Rrs}{8R^3}}$$

$$=\sqrt{\frac{6rs}{R}}\overset{\substack{Euler\\ and\\ Mitrinovic}}{\le}\sqrt{\frac{3R}{2R}.3\sqrt{3}.R}=\frac{3}{2}.\sqrt{2\sqrt{3}.R}=\frac{3}{2}.\sqrt{\sqrt{12R^2}}$$

$$\therefore\sum_{cyc}\sqrt{a\cos A}\le\frac{3.\sqrt[4]{12R^2}}{2}\ \forall\text{ acute }\Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

4130. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{r_c^2}\ge 2\sum_{cyc}\frac{\sqrt{a}}{h_a^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{r_c^2}\overset{?}{\ge}2\sum_{cyc}\frac{\sqrt{a}}{h_a^2}\Leftrightarrow\frac{(\sqrt{b}+\sqrt{c})(b+c-a)^2}{F^2}\overset{?}{\ge}2.\sum_{cyc}\frac{a^2.\sqrt{a}}{F^2}$$

$$\Leftrightarrow\sum_{cyc}\left((y+z)(y^2+z^2-x^2)^2\right)\overset{?}{\ge}2\sum_{cyc}x^5\ \left(\sqrt{a}=x,\sqrt{b}=y,\sqrt{c}=z\right)$$

$$\Leftrightarrow \sum_{cyc} x^4 y + \sum_{cyc} xy^4 \overset{?}{\geq} 2xyz\left(\sum_{cyc} xy\right) \to \text{true} \because \sum_{cyc} x^4 y + \sum_{cyc} xy^4 =$$

$$\sum_{cyc}\left(z(x^4+y^4)\right) \overset{AM-GM}{\geq} 2\sum_{cyc} x^2y^2z = 2xyz\left(\sum_{cyc} xy\right)$$

$$\therefore \sum_{cyc} \frac{\sqrt{a}+\sqrt{b}}{r_c^2} \geq 2\sum_{cyc} \frac{\sqrt{a}}{h_h^2}\ \forall\, \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4131. In ΔABC the following relationship holds:

$$(r_a+r_b)^{c^2}.(r_b+r_c)^{a^2}.(r_a+r_c)^{b^2} \leq (3R)^{9R^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$(r_a+r_b)^{c^2}.(r_b+r_c)^{a^2}.(r_a+r_c)^{b^2} \overset{Weighted\ AM-GM}{\leq}$$

$$\leq \left(\frac{c^2(r_a+r_b)+a^2(r_b+r_c)+b^2(r_a+r_c)}{(a^2+b^2+c^2)}\right)^{\sum a^2} \leq$$

$$In\ \Delta ABC\ wlog:\ a \leq b \leq c, \quad r_a \leq r_b \leq r_c$$

$$\to \{r_a+r_b \leq r_a+r_c \leq r_b+r_c$$

$$\overset{Chebyshev}{\leq} \left(\frac{2(r_a+r_b+r_c).\frac{1}{3}(a^2+b^2+c^2)}{(a^2+b^2+c^2)}\right)^{\sum a^2} =$$

$$= \left(\frac{2}{3}(4R+r)\right)^{\sum a^2} \overset{Euler}{\leq} \left(\frac{2}{3}.\frac{9R}{2}\right)^{\sum a^2} \overset{Leibniz}{\leq} (3R)^{18R^2}$$

$$Equality\ holds\ if\ a=b=c$$

4132. In acute ΔABC the following relationship holds:

$$\frac{abc}{HA \cdot HB \cdot HC} \geq 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{abc}{HA \cdot HB \cdot HC} \geq 3\sqrt{3} \Leftrightarrow \frac{2R\sin A \cdot 2R\sin B \cdot 2R\sin C}{2R\cos A \cdot 2R\cos B \cdot 2R\cos C} \geq 3\sqrt{3}$$

$$\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}\cdot\frac{\sin C}{\cos C}\ge 3\sqrt{3}$$

$$\tan A\cdot\tan B\cdot\tan C\ge 3\sqrt{3}$$

$$\tan A+\tan B+\tan C\ge 3\sqrt{3}\quad \text{(to prove)}$$

$$\tan A+\tan B+\tan C\overset{JENSEN}{\ge} 3\tan\left(\frac{A+B+C}{3}\right)=3\tan\left(\frac{\pi}{3}\right)=3\sqrt{3}$$

Equality holds for $A=B=C$.

4133. In acute ΔABC the following relationship holds:

$$\frac{HA}{a\cos A}+\frac{HB}{b\cos B}+\frac{HC}{c\cos C}\ge 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{HA}{a\cos A}+\frac{HB}{b\cos B}+\frac{HC}{c\cos C}=\sum_{cyc}\frac{HA}{a\cos A}=$$

$$=\sum_{cyc}\frac{2R\cos A}{a\cos A}=2R\sum_{cyc}\frac{1}{a}=\sum_{cyc}\frac{2R}{2R\sin A}=$$

$$=\sum_{cyc}\frac{1}{\sin A}\overset{JENSEN}{\ge}3\cdot\frac{1}{\sin\left(\frac{A+B+C}{3}\right)}=\frac{3}{\sin\frac{\pi}{3}}=\frac{3}{\frac{\sqrt{3}}{2}}=\frac{6}{\sqrt{3}}=\frac{6\sqrt{3}}{3}=2\sqrt{3}$$

Equality holds for $a=b=c$.

4134. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin B\sin C}{\sin A}\cdot\frac{b^2h_c+c^2h_b}{h_ah_b\sin B+h_ah_c\sin C}\ge\frac{24r^2}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Jenish Rijal-Nepal

$$\sum_{cyc}\frac{\sin B\sin C}{\sin A}\cdot\frac{b^2h_c+c^2h_b}{h_ah_b\sin B+h_ah_c\sin C}=$$

$$=\sum_{cyc}\frac{\frac{b}{2R}\cdot\frac{c}{2R}}{\frac{a}{2R}}\cdot\frac{b^2\cdot\frac{2\Delta}{c}+c^2\cdot\frac{2\Delta}{b}}{\frac{2\Delta}{a}\cdot\frac{2\Delta}{b}\cdot\frac{b}{2R}+\frac{2\Delta}{a}\cdot\frac{2\Delta}{c}\cdot\frac{c}{2R}}=$$

$$=\sum_{cyc}\frac{bc}{a}\cdot\frac{\frac{b^2}{c}+\frac{c^2}{b}}{\frac{2\Delta}{a}+\frac{2\Delta}{a}}=\sum_{cyc}\frac{1}{4\Delta}(b^3+c^3)=\frac{1}{2\Delta}(a^3+b^3+c^3)\underbrace{\geq}_{AM-GM}$$

$$\geq\frac{1}{2\Delta}\cdot 3abc=\frac{1}{2\cdot\frac{abc}{4R}}\cdot 3abc=6R=\frac{6R^2}{R}\underbrace{\geq}_{Mitrinovic}\frac{6(2r)^2}{R}=\frac{24r^2}{R}$$

Equality holds if and only if the triangle is equilateral.

4135. If in $\Delta ABC, = 80^\circ, B = C = 50^\circ$, $AB = AC = a$, $BC = b$ then:

$$3a^2b = b^3 + a^3\sqrt{3}$$

Proposed by Adgozel Adgozalov-Azerbaijan

Solution by Tapas Das-India

$$\frac{b}{\sin 80^\circ}=\frac{a}{\sin 50^\circ}\Rightarrow\frac{b}{\sin 80^\circ}=\frac{a}{\sin(90^\circ-40^\circ)}$$

$$\Rightarrow\frac{b}{2sin40^\circ cos40^\circ}=\frac{a}{\cos 40^\circ}\Rightarrow b=2asin40^\circ$$

$$3a^2b=3a^2\times 2asin40^\circ=6a^3sin40^\circ$$

$$b^3+a^3\sqrt{3}=8a^3\sin^3 40^\circ+a^3\sqrt{3}$$

We need to show :

$$6a^3sin40^\circ=8a^3\sin^3 40^\circ+a^3\sqrt{3}or, 6sin40^\circ-8\sin^3 40^\circ=\sqrt{3}$$

$$L.H.S=6sin40^\circ-8\sin^3 40^\circ=2(3sin40^\circ-4\sin^3 40^\circ)=$$

$$=2\sin 3.40^\circ=2\sin 120^\circ=2\times\frac{\sqrt{3}}{2}=\sqrt{3}$$

4136. In ΔABC the following relationship holds:

$$2R-r-2\sqrt{R(R-2r)}\leq min\{r_a,r_b,r_c\}\leq min\{h_a,h_b,h_c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Jenish Rijal-Nepal

$$W.L.O.G.\ let's\ assume\ that\ a \le b \le c. Then\ we\ have:$$

$$min\{r_a, r_b, r_c\} = r_a \quad and \quad min\{h_a, h_b, h_c\} = h_c$$

$$Here, a \le b \Leftrightarrow c \le b + c - a \Leftrightarrow c \le 2s - 2a \Leftrightarrow \frac{\Delta}{s-a} \le \frac{2\Delta}{c}$$

$$\Leftrightarrow r_a \le h_c \quad \therefore min\{r_a, r_b, r_c\} \le min\{h_a, h_b, h_c\}$$

$$Now, \qquad (r_b + r_c)\left(\frac{1}{r_b} + \frac{1}{r_c}\right) \overset{AM-HM}{\gtrsim} 4$$

$$Via\ identities\ \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}\ and\ r_a + r_b + r_c = 4R + r, we\ substitute:$$

$$(4R + r - r_a)\left(\frac{1}{r} - \frac{1}{r_a}\right) \ge 4 \Leftrightarrow (4R + r - r_a)(r_a - r) \ge 4rr_a$$

$$\Leftrightarrow r_a^2 - 2(2R - r)r_a + r(4R + r) \le 0$$

$$\Leftrightarrow \left(r_a - 2R + r + 2\sqrt{R(R-2r)}\right)\left(r_a - 2R + r - 2\sqrt{R(R-2r)}\right) \le 0$$

$$\therefore 2R - r - 2\sqrt{R(R-2r)} \le r_a \le 2R - r + 2\sqrt{R(R-2r)}$$

$$Extracting\ the\ lower\ bound\ gives:$$

$$2R - r - 2\sqrt{R(R-2r)} \le r_a = min\{r_a, r_b, r_c\}$$

$$Therefore, \qquad 2R - r - 2\sqrt{R(R-2r)} \le min\{r_a, r_b, r_c\} \le min\{h_a, h_b, h_c\}$$

$$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$$

4137.In any Δ ABC the following relationship holds :

$$\sqrt{\frac{2R}{r}} \ge \sqrt{\left(\frac{r_a}{p_a}\right)\left(\sqrt{\frac{g_a}{l_a}}\right)} + \sqrt{\left(\frac{p_a}{r_a}\right)\left(\sqrt{\frac{l_a}{g_a}}\right)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality} \Rightarrow g_a \le AI + r \overset{?}{\le} l_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} \frac{a}{(b+c)\sin\frac{A}{2}} + \frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\le} a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\le} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \overset{?}{\le} 1 \to \text{true} \therefore l_a \ge g_a \to \text{(i)}$$

$$\text{Now, } m_a n_a \overset{?}{\ge} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow} \left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\ge}$$

$$\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2 + \frac{(b-c)^4}{324} +$$

$$\frac{(b-c)^2}{9}\cdot\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\ge} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a).\frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) +$$

$$\left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\ge} 0 \ (\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} +$$

$$\frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3\right)}{36a(2s+a)^2} +$$

$$\frac{(s-a)\left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\ge} 0 \to \text{true (strict inequality)} \therefore m_a n_a \ge p_a^2 + \frac{(b-c)^2}{18} \ge p_a^2 \Rightarrow m_a n_a \ge p_a^2 \to (1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\ge} m_a l_a \to (2) \therefore (1)\bullet(2) \Rightarrow (m_a n_a)(n_a g_a) \ge p_a^2.m_a l_a$$

$$\Rightarrow n_a \ge p_a\cdot\sqrt{\frac{l_a}{g_a}} \Rightarrow \boxed{t \le n_a}\left(t = p_a\cdot\sqrt{\frac{l_a}{g_a}}\right)$$

$$\Rightarrow\left(\sqrt{\left(\frac{r_a}{p_a}\right)\left(\sqrt{\frac{g_a}{l_a}}\right)}+\sqrt{\left(\frac{p_a}{r_a}\right)\left(\sqrt{\frac{l_a}{g_a}}\right)}\right)^2 = \frac{r_a}{t}+\frac{t}{r_a}+2 = \frac{r_a^2+t^2}{r_a\cdot p_a\cdot\sqrt{\frac{l_a}{g_a}}}+2 \le$$

$$\frac{r_a^2+n_a^2}{r_a\cdot h_a\cdot\sqrt{\frac{l_a}{g_a}}}+2 \overset{\substack{\text{via (i)}\\ \text{and}\\ \text{via Bogdan Fustei}}}{\le} \frac{s^2\tan^2\frac{A}{2}+s^2-2r_ah_a}{r_ah_a}+2 = \frac{2s^2\sec^2\frac{A}{2}}{\frac{r}{R}\cdot s^2.\sec^2\frac{A}{2}} = \frac{2R}{r}$$

$$\Rightarrow\sqrt{\frac{2R}{r}} \ge \sqrt{\left(\frac{r_a}{p_a}\right)\left(\sqrt{\frac{g_a}{l_a}}\right)}+\sqrt{\left(\frac{p_a}{r_a}\right)\left(\sqrt{\frac{l_a}{g_a}}\right)}\ \forall\,\Delta\,ABC,''='' \text{ iff } b = c\ (QED)$$

4138. In any Δ ABC with $a \le b \le c$, the following relationships holds :

$$\frac{\sqrt{3}}{2}a \le g_b \le p_b \le \frac{\sqrt{3}}{2}c\,;\ \text{and } p_c \le \frac{\sqrt{3}}{2}b \le g_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x,$ $c = x+y$ and $s = x+y+z$ and also, $a \le b \le c \Rightarrow y+z \le z+x \le x+y$

$$\Rightarrow \boxed{x \ge y \ge z} \text{ and now, } \frac{3}{4}a^2 \overset{?}{\le} g_b^2 \overset{\text{via Bogdan Fustei}}{\Leftrightarrow} \frac{3}{4}(y+z)^2 \overset{?}{\le} y^2+\frac{4xyz}{z+x}$$

$$\Leftrightarrow xy^2+10xyz+y^2z \overset{?}{\ge} 3xz^2+6yz^2+3z^3$$

$$\Leftrightarrow 6yz(x-z)+3xz(y-z)+z(xy-z^2) \overset{?}{\ge} 0 \to \text{true} \because x \ge y \ge z \therefore \boxed{\boxed{\frac{\sqrt{3}}{2}a \le g_b}}$$

$$\text{Also, } p_b^2 \overset{?}{\le} \frac{3}{4}c^2 \overset{\substack{\text{via Bogdan Fustei}\\ \text{and}\\ \text{Mohamed Amine Ajiba}}}{\Leftrightarrow} s(s-b)+\frac{s(3s+b)(c-a)^2}{(2s+b)^2} \overset{?}{\le} \frac{3}{4}c^2$$

$$\Leftrightarrow (x+y+z)\left(y+\frac{(3(x+y+z)+z+x)(z-x)^2}{(2(x+y+z)+z+x)^2}\right) \overset{?}{\le} \frac{3}{4}(x+y)^2$$

$$\Leftrightarrow 11x^4+26x^3y+54x^3z+15x^2y^2+64x^2yz+59x^2z^2 \overset{?}{\ge}$$
$$4xy^3+18xy^2z+26xyz^2+4y^4+28y^3z+69y^2z^2+64yz^3+16z^4$$
$$\Leftrightarrow 18xyz(x-y)+28yz(x^2-y^2)+18xyz(x-z)+59z^2(x^2-y^2)+$$

$$54z(x^3-yz^2)+8xy(x^2-z^2)+4xy(x^2-y^2)+4y(x^3-y^3)+10y(x^3-yz^2)+$$

$$10y(x^2y-z^3)+11(x^4-z^4)+5(x^2y^2-z^4)\overset{?}{\geq}0 \to \text{true} \because x\geq y\geq z$$

$$\therefore \boxed{p_b\leq\frac{\sqrt{3}}{2}c}$$

$$\text{Furthermore, } p_c^2\overset{?}{\leq}\frac{3}{4}b^2 \overset{\substack{\text{via Bogdan Fustei}\\ \text{and}\\ \text{Mohamed Amine Ajiba}}}{\Leftrightarrow} s(s-c)+\frac{s(3s+c)(a-b)^2}{(2s+c)^2}\overset{?}{\leq}\frac{3}{4}b^2$$

$$\Leftrightarrow (x+y+z)\left(z+\frac{(3(x+y+z)+x+y)(x-y)^2}{(2(x+y+z)+x+y)^2}\right)\overset{?}{\leq}\frac{3}{4}(z+x)^2$$

$$\Leftrightarrow 11x^4+26x^3z+54x^3y+15x^2z^2+64x^2yz+59x^2y^2\overset{?}{\geq}$$

$$4xz^3+18xyz^2+26xy^2z+4z^4+28yz^3+69y^2z^2+64y^3z+16y^4$$

$$\Leftrightarrow 64yz(x^2-y^2)+26xz(x^2-y^2)+59y^2(x^2-z^2)+28y(x^3-z^3)+$$

$$18xy(x^2-z^2)+10z^2(x^2-y^2)+8y(x^3-y^3)+8(x^4-y^4)+4z^2(x^2-z^2)+$$

$$xz^2(x-z)+3x(x^3-z^3)\overset{?}{\geq}0\to \text{true} \because x\geq y\geq z \therefore \boxed{p_c\leq\frac{\sqrt{3}}{2}b} \text{ and again,}$$

$$\frac{3}{4}b^2\overset{?}{\leq}g_a^2 \overset{\text{via Bogdan Fustei}}{\Leftrightarrow} \frac{3}{4}(z+x)^2\overset{?}{\leq}x^2+\frac{4xyz}{y+z}$$

$$\Leftrightarrow x^2y+10xyz+x^2z\overset{?}{\geq}3yz^2+6xz^2+3z^3$$

$$\Leftrightarrow 6xz(y-z)+3yz(x-z)+z(xy-z^2)\overset{?}{\geq}0\to \text{true} \because x\geq y\geq z$$

$$\therefore \boxed{\frac{\sqrt{3}}{2}b\leq g_a} \text{ and finally, } g_b^2\overset{?}{\leq}p_b^2 \overset{\substack{\text{via Bogdan Fustei}\\ \text{and}\\ \text{Mohamed Amine Ajiba}}}{\Leftrightarrow}$$

$$s(s-b)-\frac{(s-b)(c-a)^2}{b}\overset{?}{\leq}s(s-b)+\frac{s(3s+b)(c-a)^2}{(2s+b)^2}\to \text{evidently true}$$

$$\because s>b \therefore \boxed{g_b\leq p_b} \text{ and so, } \frac{\sqrt{3}}{2}a\leq g_b\leq p_b\leq\frac{\sqrt{3}}{2}c\text{; and } p_c\leq\frac{\sqrt{3}}{2}b\leq g_a$$

$$\forall\,\Delta\,ABC,\text{"="} \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

4139. **In any Δ ABC the following relationship holds :**

$$2R-r-2\sqrt{R(R-2r)}\leq\min\{r_a,r_b,r_c\}\leq\min\{h_a,h_b,h_c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_a+r=\frac{r.p}{p-a}+\frac{r.p}{p}=\frac{r(b+c)}{p-a}\ (p\to \text{semi}-\text{perimeter})$$

$$= \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \therefore r_a + r \overset{①}{=} 4Rsc\left(c = \cos\frac{B-C}{2}, s = \sin\frac{A}{2}\right)$$

$$\textit{and again}, 2R - 2\sqrt{R(R-2r)} \overset{②}{=} 2R - 2R.\sqrt{1 - 4sc + 4s^2}$$

$$\text{Now}, ① \textit{ and } ② \Rightarrow r_a \overset{?}{\geq} 2R - r - 2\sqrt{R(R-2r)}$$

$$\Leftrightarrow 4Rsc \overset{?}{\geq} 2R - 2R.\sqrt{1 - 4sc + 4s^2} \Leftrightarrow \sqrt{1 - 4sc + 4s^2} \boxed{\underset{(*)}{\overset{?}{\geq}}} 1 - 2sc \textit{ and if}$$

$$1 - 2sc < 0, \textit{then}: (**) \text{ is trivially true and when}: 1 - 2sc \geq 0, \text{then}:$$

$$(*) \Leftrightarrow 1 - 4sc + 4s^2 \overset{?}{\geq} 4s^2c^2 - 4sc + 1 \Leftrightarrow 4s^2(1 - c^2) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because c^2 = \cos^2\frac{B-C}{2} \leq 1 \Rightarrow (*) \text{ is true } \forall\, \Delta\, ABC$$

$$\therefore r_a \geq 2R - r - 2\sqrt{R(R-2r)} \textit{ and similarly } r_b, r_c \geq 2R - r - 2\sqrt{R(R-2r)}$$

$$\textit{and so}, \boxed{\boxed{\min\{r_a, r_b, r_c\} \geq 2R - r - 2\sqrt{R(R-2r)}}} \textit{ and now},$$

we seggregate into the following cases :

$\boxed{\text{Case 1}}$ $a \leq b \leq c$ and then : $\min\{r_a, r_b, r_c\} \overset{?}{\leq} \min\{h_a, h_b, h_c\} \Leftrightarrow r_a \overset{?}{\leq} h_c$

$$\Leftrightarrow b + c - a \overset{?}{\geq} c \Leftrightarrow b \overset{?}{\geq} a \rightarrow \text{true}$$

$\boxed{\text{Case 2}}$ $a \leq c \leq b$ and then : $\min\{r_a, r_b, r_c\} \overset{?}{\leq} \min\{h_a, h_b, h_c\} \Leftrightarrow r_a \overset{?}{\leq} h_b$

$$\Leftrightarrow b + c - a \overset{?}{\geq} b \Leftrightarrow c \overset{?}{\geq} a \rightarrow \text{true}$$

$\boxed{\text{Case 3}}$ $b \leq c \leq a$ and then : $\min\{r_a, r_b, r_c\} \overset{?}{\leq} \min\{h_a, h_b, h_c\} \Leftrightarrow r_b \overset{?}{\leq} h_a$

$$\Leftrightarrow c + a - b \overset{?}{\geq} a \Leftrightarrow c \overset{?}{\geq} b \rightarrow \text{true}$$

$\boxed{\text{Case 4}}$ $b \leq a \leq c$ and then : $\min\{r_a, r_b, r_c\} \overset{?}{\leq} \min\{h_a, h_b, h_c\} \Leftrightarrow r_b \overset{?}{\leq} h_c$

$$\Leftrightarrow c + a - b \overset{?}{\geq} c \Leftrightarrow a \overset{?}{\geq} b \rightarrow \text{true}$$

$\boxed{\text{Case 5}}$ $c \leq a \leq b$ and then : $\min\{r_a, r_b, r_c\} \overset{?}{\leq} \min\{h_a, h_b, h_c\} \Leftrightarrow r_c \overset{?}{\leq} h_b$

$$\Leftrightarrow a + b - c \overset{?}{\geq} b \Leftrightarrow a \overset{?}{\geq} c \rightarrow \text{true}$$

$\boxed{\text{Case 6}}$ $c \leq b \leq a$ and then : $\min\{r_a, r_b, r_c\} \overset{?}{\leq} \min\{h_a, h_b, h_c\} \Leftrightarrow r_c \overset{?}{\leq} h_a$

$$\Leftrightarrow a + b - c \overset{?}{\geq} a \Leftrightarrow b \overset{?}{\geq} c \rightarrow \text{true} \therefore \text{combining all cases},$$

$$\boxed{\boxed{\min\{r_a, r_b, r_c\} \leq \min\{h_a, h_b, h_c\}}} \textit{ and so},$$

$$2R - r - 2\sqrt{R(R-2r)} \leq \min\{r_a, r_b, r_c\} \leq \min\{h_a, h_b, h_c\} \,\forall\, \Delta\, ABC,$$
$$'' = '' \text{ iff } \Delta\, ABC \text{ is isosceles (QED)}$$

4140. In any Δ ABC the following relationship holds :

$$\max\{n_a, n_b, n_c\} \le \max\{r_a, r_b, r_c\} \le 2R - r + 2\sqrt{R(R-2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_a + r = \frac{r.p}{p-a} + \frac{r.p}{p} = \frac{r(b+c)}{p-a} \ (p \to \text{semi}-\text{perimeter})$$

$$= \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \therefore r_a + r \overset{(1)}{=} 4Rsc$$

$$\left(\text{where } c = \cos\frac{B-C}{2}, s = \sin\frac{A}{2}\right) \text{and again,}$$

$$2R + 2\sqrt{R(R-2r)} \overset{(2)}{=} 2R + 2R.\sqrt{1-4sc+4s^2}$$

$$\text{Now, } (1) \text{ and } (2) \Rightarrow r_a \overset{?}{\le} 2R - r + 2\sqrt{R(R-2r)}$$

$$\Leftrightarrow 4Rsc \overset{?}{\le} 2R + 2R.\sqrt{1-4sc+4s^2} \Leftrightarrow 2sc - 1 \boxed{\overset{?}{\le}(*)} \sqrt{1-4sc+4s^2} \text{ and if}$$

$$2sc - 1 < 0, \text{then} : (*) \text{ is trivially true and when} : 2sc - 1 \ge 0 \text{ and then} :$$

$$(*) \Leftrightarrow 4s^2c^2 - 4sc + 1 \overset{?}{\le} 1 - 4sc + 4s^2 \Leftrightarrow 4s^2(c^2-1) \le 0 \to \text{true}$$

$$\because c^2 = \cos^2\frac{B-C}{2} \le 1 \Rightarrow (*) \text{ is true } \forall \Delta ABC$$

$$\therefore r_a \le 2R - r + 2\sqrt{R(R-2r)} \text{ and similarly}$$

$$r_b, r_c \le 2R - r + 2\sqrt{R(R-2r)} \text{ and so,}$$

$$\boxed{\boxed{\max\{r_a, r_b, r_c\} \le 2R - r + 2\sqrt{R(R-2r)}}} \text{ and now, } n_a^2 \overset{\text{Bogdan Fustei}}{=}$$

$$s^2 - \frac{r}{R}.s^2.\sec^2\frac{A}{2} \therefore \text{if } a \le b, \text{then, as } \sec^2\frac{A}{2} \le \sec^2\frac{B}{2}, \text{hence } n_a^2 \ge n_b^2 \to (1)$$

and we seggregate into the following cases :

$$\boxed{\text{Case 1}}\ a \le b \le c \text{ and then} : \max\{n_a, n_b, n_c\} \overset{?}{\le} \max\{r_a, r_b, r_c\} \overset{\text{via }(1)}{\Leftrightarrow} n_a^2 \overset{?}{\le} r_c^2$$

$$\overset{\text{Bogdan Fustei}}{\Leftrightarrow} s(s-a) + \frac{s}{a}(b-c)^2 \overset{?}{\le} \frac{s(s-a)(s-b)(s-c)}{(s-c)^2}$$

$$\Leftrightarrow \frac{xy}{z} \overset{?}{\ge} x + \frac{(y-z)^2}{y+z} \ (x = s-a, y = s-b, z = s-c)$$

$$\Leftrightarrow (y-z)\left(xy + z^2 + z(x-y)\right) \overset{?}{\ge} 0 \to \text{true} \because a \le b \le c \Rightarrow x \ge y \ge z$$

$$\boxed{\text{Case 2}}\ a \le c \le b \text{ and then} : \max\{n_a, n_b, n_c\} \overset{?}{\le} \max\{r_a, r_b, r_c\} \overset{\text{via }(1)}{\Leftrightarrow} n_a^2 \overset{?}{\le} r_b^2$$

$$\Leftrightarrow (z-y)\left(xz + y^2 + y(x-z)\right) \overset{?}{\ge} 0 \to \text{true} \because a \le c \le b \Rightarrow x \ge z \ge y$$

$\boxed{\text{Case 3}}$ $b \le c \le a$ and then : $max\{n_a, n_b, n_c\} \overset{?}{\le} max\{r_a, r_b, r_c\} \overset{\text{via (1)}}{\Leftrightarrow} n_b^2 \overset{?}{\le} r_a^2$

$$\Leftrightarrow (z - x)\left(yz + x^2 + x(y - z)\right) \overset{?}{\ge} 0 \to \text{true} \because b \le c \le a \Rightarrow y \ge z \ge x$$

$\boxed{\text{Case 4}}$ $b \le a \le c$ and then : $max\{n_a, n_b, n_c\} \overset{?}{\le} max\{r_a, r_b, r_c\} \overset{\text{via (1)}}{\Leftrightarrow} n_b^2 \overset{?}{\le} r_c^2$

$$\Leftrightarrow (x - z)\left(xy + z^2 + z(y - x)\right) \overset{?}{\ge} 0 \to \text{true} \because b \le a \le c \Rightarrow y \ge x \ge z$$

$\boxed{\text{Case 5}}$ $c \le a \le b$ and then : $max\{n_a, n_b, n_c\} \overset{?}{\le} max\{r_a, r_b, r_c\} \overset{\text{via (1)}}{\Leftrightarrow} n_c^2 \overset{?}{\le} r_b^2$

$$\Leftrightarrow (x - y)\left(zx + y^2 + y(z - x)\right) \overset{?}{\ge} 0 \to \text{true} \because c \le a \le b \Rightarrow z \ge x \ge y$$

$\boxed{\text{Case 6}}$ $c \le b \le a$ and then : $max\{n_a, n_b, n_c\} \overset{?}{\le} max\{r_a, r_b, r_c\} \overset{\text{via (1)}}{\Leftrightarrow} n_c^2 \overset{?}{\le} r_a^2$

$$\Leftrightarrow (y - x)\left(yz + x^2 + x(z - y)\right) \overset{?}{\ge} 0 \to \text{true} \because c \le b \le a \Rightarrow z \ge y \ge x$$

$\therefore$ combining *all* cases, $\boxed{\boxed{max\{n_a, n_b, n_c\} \le max\{r_a, r_b, r_c\}}}$ and so,

$$max\{n_a, n_b, n_c\} \le max\{r_a, r_b, r_c\} \le 2R - r + 2\sqrt{R(R - 2r)}\ \forall\, \Delta\, ABC,$$
$$'' = ''\text{ iff } \Delta\, ABC \text{ is isosceles (QED)}$$

4141. In ΔABC the following relationship holds:

$$2R - r - 2\sqrt{R(R - 2r)} \le min\{r_a, r_b, r_c\} \le min\{h_a, h_b, h_c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Jenish Rijal-Bardiya-Nepal

$$W.L.O.G.\text{ let's assume that } a \le b \le c. \text{ Then we have:}$$

$$min\{r_a, r_b, r_c\} = r_a \quad and \quad min\{h_a, h_b, h_c\} = h_c$$

$$Here, a \le b \Leftrightarrow c \le b + c - a \Leftrightarrow c \le 2s - 2a \Leftrightarrow \frac{\Delta}{s - a} \le \frac{2\Delta}{c}$$

$$\Leftrightarrow r_a \le h_c \quad \therefore min\{r_a, r_b, r_c\} \le min\{h_a, h_b, h_c\}$$

$$Now, \quad (r_b + r_c)\left(\frac{1}{r_b} + \frac{1}{r_c}\right) \overset{AM-HM}{\ge} 4$$

$$Via\ identities\ \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}\ and\ r_a + r_b + r_c = 4R + r, we\ substitute:$$

$$(4R + r - r_a)\left(\frac{1}{r} - \frac{1}{r_a}\right) \ge 4 \Leftrightarrow (4R + r - r_a)(r_a - r) \ge 4rr_a$$

$$\Leftrightarrow r_a^2 - 2(2R - r)r_a + r(4R + r) \le 0$$

$$\Leftrightarrow \left(r_a - 2R + r + 2\sqrt{R(R-2r)}\right)\left(r_a - 2R + r - 2\sqrt{R(R-2r)}\right) \le 0$$

$\therefore\ 2R - r - 2\sqrt{R(R-2r)} \le r_a \le 2R - r + 2\sqrt{R(R-2r)}$

Extracting the lower bound gives:

$$2R - r - 2\sqrt{R(R-2r)} \le r_a = min\{r_a, r_b, r_c\}$$

Therefore, $\quad 2R - r - 2\sqrt{R(R-2r)} \le min\{r_a, r_b, r_c\} \le min\{h_a, h_b, h_c\}$

Equality holds if and only if the triangle is equilateral.

4142. In ΔABC the following relationship holds:

$$min\{m_a, m_b, m_c\} \le \frac{1}{2}\sqrt{s^2 - 4Rr + 17r^2} \le \sqrt{s^2 - 7Rr - 4r^2} \le max\{m_a, m_b, m_c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Jenish Rijal-Bardiya--Nepal

W.L.O.G. let's assume that $a \le b \le c$. *Then we have*: $m_a \ge m_b \ge m_c$

$$min\{m_a, m_b, m_c\} = m_c \text{ and } max\{m_a, m_b, m_c\} = m_a$$

Also let $(a, b, c) = (y + z, z + x, x + y)$ *which gives* $x \ge y \ge z$ *with* $x, y, z > 0$.

Here, $(s^2 - 16Rr + 5r^2) + 8r(R - 2r) \overset{Gerretsen+Euler}{\ge} 0 \Leftrightarrow s^2 - 8Rr - 11r^2 \ge 0$

$$\Leftrightarrow 4(s^2 - 7Rr - 4r^2) - (s^2 - 4Rr + 17r^2) \ge 0 \Leftrightarrow \frac{1}{4}(s^2 - 4Rr + 17r^2) \le s^2 - 7Rr - 4r^2$$

$$\therefore \frac{1}{2}\sqrt{s^2 - 4Rr + 17r^2} \le \sqrt{s^2 - 7Rr - 4r^2}$$

i.) Let's prove that: $min\{m_a, m_b, m_c\} = m_c \le \frac{1}{2}\sqrt{s^2 - 4Rr + 17r^2}$

Squaring $m_c \le \frac{1}{2}\sqrt{s^2 - 4Rr + 17r^2}$ *and substituting* $m_c^2 = s^2 - r^2 - 4Rr - \frac{3}{4}c^2$ *yields*:

$$s^2 - 4Rr - 7r^2 \le c^2 \overset{s^2 - 4Rr = \frac{a^2+b^2+c^2}{2} + r^2}{\Leftrightarrow} a^2 + b^2 - c^2 \le 12r^2$$

$\Leftrightarrow z(x + y + z)^2 \le xy(x + y + 7z) \Leftrightarrow (x - z)(y - z)(x + y + 7z) + 4z^2[(x - z) + (y - z)] \ge 0$ *which is true because* $x \ge z$.

ii.) Let's prove that: $\sqrt{s^2-7Rr-4r^2}\le max\{m_a,m_b,m_c\}=m_a$

Squaring $\sqrt{s^2-7Rr-4r^2}\le m_a$ *and substituting* $m_a^2=s^2-r^2-4Rr-\frac{3}{4}a^2$ *yields:*

$$s^2-7Rr-4r^2\le s^2-r^2-4Rr-\frac{3}{4}a^2 \quad \Leftrightarrow \quad a^2\le 4Rr+4r^2$$

$$\Leftrightarrow\ (y+z)^2\le\frac{(x+y)(y+z)(z+x)+4xyz}{x+y+z}\Leftrightarrow x^2+\frac{4xyz}{y+z}\ge y^2+yz+z^2.\ Also,$$

$$x^2+\frac{4xyz}{y+z}\overset{x\ge y}{\ge} y^2+\frac{4y^2z}{y+z}.Thus\ it\ suffices\ to\ prove\ that: y^2+\frac{4y^2z}{y+z}\ge y^2+yz+z^2$$

$$\Leftrightarrow\frac{4y^2z}{y+z}\ge yz+z^2\Leftrightarrow 4y^2\ge(y+z)^2\Leftrightarrow y\ge z\ which\ is\ true\ because\ y\ge z.$$

$$min\{m_a,m_b,m_c\}\le\frac{1}{2}\sqrt{s^2-4Rr+17r^2}\le\sqrt{s^2-7Rr-4r^2}\le max\{m_a,m_b,m_c\}$$

Equality holds if and only if the triangle is equilateral.

4143. In ΔABC the following relationship holds:

$$m_a^2-w_a^2\le\sqrt{\frac{2}{27}}|b^2-c^2|$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$m_a^2=s(s-a)+\frac{1}{4}(b-c)^2,w_a^2=s(s-a)-\frac{s(s-a)}{(b+c)^2}(b-c)^2$$

$$m_a^2-w_a^2=\frac{(b-c)^2\left((b+c)^2+4s(s-a)\right)}{4(b+c)^2}=\frac{(b-c)^2(2(b+c)^2-a^2)}{4(b+c)^2}$$

$$let\ x=b+c,y=|b-c|\ then\ |b^2-c^2|=xy,\qquad m_a^2-w_a^2=\frac{y^2(2x^2-a^2)}{4x^2}$$

$$now,\frac{m_a^2-w_a^2}{|b^2-c^2|}=\frac{y(2x^2-a^2)}{4x^3}=\frac{\frac{y}{x}\left(2-\frac{a^2}{x^2}\right)}{4}=\frac{t(2-u^2)}{4}\overset{t<u}{\le}\frac{t(2-t^2)}{4}$$

$$\left(t=\frac{y}{x}=\frac{|b-c|}{b+c},u=\frac{a}{b+c}\ and\ clearly,0<t<u<1\ as\ |b-c|<a\right)$$

$$let\ f(t)=\frac{t(2-t^2)}{4}\ then\ f'(t)=\frac{2-3t^2}{4},f''(t)=-\frac{3t}{2}$$

$$now\ f'(t)=0\Rightarrow t=\sqrt{\frac{2}{3}}\ \&\ f''(t)\ at\ t=\sqrt{\frac{2}{3}}=-\frac{3}{2}\sqrt{\frac{2}{3}}<0\ so$$

$$f_{Max}=\frac{\sqrt{\frac{2}{3}}\left(2-\frac{2}{3}\right)}{4}=\sqrt{\frac{2}{27}}$$

$$\frac{m_a^2-w_a^2}{|b^2-c^2|}=\frac{y(2x^2-a^2)}{4x^3}=\frac{\frac{y}{x}\left(2-\frac{a^2}{x^2}\right)}{4}=\frac{t(2-u^2)}{4}\overset{t<u}{\leq}\frac{t(2-t^2)}{4}=f(t)\leq\sqrt{\frac{2}{27}}$$

$$m_a^2-w_a^2\leq\sqrt{\frac{2}{27}}|b^2-c^2|$$

Equality holds for an equilateral triangle.

4144. **In any Δ ABC the following relationship holds :**

$$\sqrt{w_an_a}\leq\sqrt{\frac{2R}{r}-3}\cdot h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a,y=s-b,z=s-c$ and then : $a=y+z,b=z+x,$ $c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$ and then, we have the following set "S" of relations : $y^2+z^2=x^2(m^2-2n)$, $y^3+z^3=x^3(m^3-3nn),y^4+z^4=x^4((m^2-2n)^2-2n^2)$,

$y^5+z^5=x^5\left(m((m^2-2n)^2+n^2-nm^2)\right)$, and now, $\sqrt{w_an_a}\overset{?}{\leq}\sqrt{\frac{2R}{r}-3}.h_a$

$$\overset{\text{Bogdan Fustei}}{\Leftrightarrow}\frac{4bc}{(b+c)^2}.s(s-a)\left(s(s-a)+\frac{s}{a}.(b-c)^2\right)\overset{?}{\leq}$$
$$\left(\frac{(y+z)(z+x)(x+y)}{2xyz}-3\right)^2.\frac{16s^2(s-a)^2(s-b)^2(s-c)^2}{a^4}$$

$$\Leftrightarrow\left((y+z)(z+x)(x+y)-6xyz\right)^2(2x+y+z)^2\overset{?}{\geq}$$
$$x(y+z)^3(z+x)(x+y)(x(y+z)+(y-z)^2)$$
$$\Leftrightarrow 4x^6(y+z)^2+12x^5(y^3+z^3)-12x^5.yz(y+z)+12x^4(y^4+z^4)-$$
$$40x^4.yz(y^2+z^2)+40x^4.y^2z^2+4x^3(y^5+z^5)-20x^3.yz(y^3+z^3)+$$

$$16x^3.y^2z^2(y+z)+x^2.yz(y^4+z^4)-4x^2.y^2z^2(y^2+z^2)-10x^2.y^3z^3+$$

$$xyz(y^5+z^5)+x.y^2z^2(y^3+z^3)-2x.y^3z^3(y+z)+y^2z^2(y+z)^4 \boxed{\overset{?}{\underset{(1)}{\geq}}} 0 \text{ \& via set of}$$

relations "S", in order to prove (1), it suffices to prove, following simplification :

$$\overbrace{(m^4-4m^3-8m^2+96m+144)}^{\Omega_1}n^2+\overbrace{(m^5+m^4-40m^3-88m^2-48m)}^{\Omega_2}n+$$

$$\overbrace{(4m^5+12m^4+12m^3+4m^2)}^{\Omega_3} \boxed{\overset{?}{\underset{(2)}{\geq}}} 0; \text{We have}: m^4-4m^3-8m^2+96m+144=$$

$$\frac{(2m-9)^2(2m+5)^2}{16}+\frac{168m^2+816m+339}{16}>0 \therefore (2) \text{ is trivially true when}:$$

$$m^5+m^4-40m^3-88m^2-48m\geq 0 \text{ \& when}: m^5+m^4-40m^3-88m^2-48m$$

$$<0, \text{then}: (m-7)(m^3+8m^2+16m+24)+120<0$$

$$\Rightarrow (m-7)(m^3+8m^2+16m+24)<-120<0\Rightarrow m<7\rightarrow (m) \text{ and } \because$$

LHS of (2) is a quadratic polynomial in "n" with discriminant,

$$\boxed{\delta}=\Omega_2^2-4\Omega_1\Omega_3=m^4(m+1)^2(m^4-16m^3-32m^2+96m+192) \therefore \text{if } \delta\leq 0,$$

then : (2) is trivially true and when : $\delta>0$, then :

$$m^4-16m^3-32m^2+96m+192>0\Rightarrow$$

$$(m^3-13m^2-71m-117)(m-3)-159>0\Rightarrow$$

$$(m^3-13m^2-71m-117)(m-3)>159>0 \text{ and } \because m^3-13m^2-71m-117=$$

$$m^2(m-7)-6m^2-71m-117\overset{\text{via }(m)}{<}0 \therefore m<3\rightarrow (n)$$

$$\text{Now, since } n\overset{\text{AM-GM}}{\leq}\frac{m^2}{4} \therefore \text{in order to prove (2), it suffices to prove}:$$

$$\Omega_1.\frac{m^2}{2}\overset{?}{\leq}-\Omega_2-\sqrt{\delta}\Leftrightarrow -2\Omega_2-\Omega_1m^2\overset{?}{\geq}2\sqrt{\delta}$$

$$\Leftrightarrow m(-m^5+2m^4+6m^3-16m^2+32m+96)\boxed{\overset{?}{\underset{(3)}{\geq}}}2\sqrt{\delta}$$

$$\text{We have}: m^5-2m^4-6m^3+16m^2-32m-96=$$

$$m^4(m-3)+m^3(m-3-3)+16m(m-3)+16(m-3-3)\overset{\text{via }(n)}{<}0$$

$$\Rightarrow m(-m^5+2m^4+6m^3-16m^2+32m+96)>0 \text{ and so,}$$

in order to prove (3), it suffices to prove :

$$m^2(-m^5+2m^4+6m^3-16m^2+32m+96)^2\overset{?}{\geq}$$

$$4m^4(m+1)^2(m^4-16m^3-32m^2+96m+192)\Leftrightarrow$$

$$m^2(m+2)^2(m-2)^2\left((2m+5)^2\left(\frac{(4m^2-18m)^2+232m^2-600m+1317}{64}\right)+\frac{13236m+3939}{64}\right)$$

$$\overset{?}{\geq} 0 \to \text{true} \because m > 0 \text{ and } \because \text{discriminant of } 232m^2 - 600m + 1317 < 0$$

$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true} \therefore \sqrt{w_a n_a} \leq \sqrt{\frac{2R}{r} - 3}.h_a \ \forall \Delta ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4145. In any ΔABC the following relationship holds :

$$\sqrt{g_a n_a} \leq \sqrt{\frac{7R}{6r} - \frac{4}{3}}.h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$ $c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$, $y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2),$

$$y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right), \text{and now}, \sqrt{g_a n_a} \overset{?}{\leq} \sqrt{\frac{7R}{6r} - \frac{4}{3}}.h_a$$

$$\overset{\text{Bogdan Fustei}}{\Leftrightarrow} s^2(s-a)^2 + \frac{4s(s-a)(s-b)(s-c)(b-c)^2}{a^2} \overset{?}{\leq}$$

$$\left(\frac{7}{6}.\frac{(y+z)(z+x)(x+y)}{4xyz} - \frac{4}{3}\right)^2.\frac{16s^2(s-a)^2(s-b)^2(s-c)^2}{a^4} \Leftrightarrow$$

$$\frac{(y+z)^2.x(x+y+z) + 4yz(y-z)^2}{(y+z)^2} \overset{?}{\leq} \frac{(7(y+z)(z+x)(x+y) - 32xyz)^2}{36x^2y^2z^2}.\frac{x(x+y+z).y^2z^2}{(y+z)^4}$$

$$\Leftrightarrow 49x^5(y+z)^2 + 147x^4(y^3+z^3) - 7x^4yz(y+z) + 111x^3(y^4+z^4) -$$
$$354x^3.yz(y^2+z^2) + 94x^3.y^2z^2 + 13x^2(y^5+z^5) - 187x^2.yz(y^3+z^3) -$$
$$50x^2.y^2z^2(y+z) - 46x.yz(y^4+z^4) - 7x.y^2z^2(y^2+z^2) + 78x.y^3z^3 +$$
$$49y^2z^2(y^3+z^3) \boxed{\underset{①}{\overset{?}{\geq}}} 0 \text{ and via set of relations "S"},$$

in order to prove ①, it suffices to prove, following simplification :

$$\overbrace{(49m^3 + 177m^2 + 576m + 1024)}^{\Omega_1} n^2 - \overbrace{(46m^4 + 252m^3 + 798m^2 + 448m)}^{\Omega_2} n +$$
$$\overbrace{(13m^5 + 111m^4 + 147m^3 + 49m^2)}^{\Omega_3} \boxed{\underset{②}{\overset{?}{\geq}}} 0$$

Now, LHS of ② is a quadratic polynomial in "n" with discriminant,

$\boxed{\delta} = \Omega_2^2 - 4\Omega_1\Omega_3 = 432m^4(-m^4 - 18m^3 - m^2 + 48m + 80)$ and when :

$\delta \le 0$, then : ② is trivially true and when : $\delta > 0$,

then : $m^4 + 18m^3 + m^2 - 48m - 80 < 0$

$$\Rightarrow \frac{(729m^3 + 14661m^2 + 31680m + 31888)(9m - 19)}{6561} < -\frac{80992}{6561} < 0$$

$$\Rightarrow m < \frac{19}{9} \to (m)$$

Now, since $n \overset{AM-GM}{\le} \frac{m^2}{4}$ ∴ in order to prove ②, it suffices to prove :

$$\Omega_1 . \frac{m^2}{2} \overset{?}{\le} \Omega_2 - \sqrt{\delta} \Leftrightarrow 2\Omega_2 - \Omega_1 m^2 \overset{?}{\ge} 2\sqrt{\delta}$$

$$\Leftrightarrow m(-49m^4 - 85m^3 - 72m^2 + 572m + 896) \boxed{\underset{③}{\overset{?}{\ge}}} 2\sqrt{\delta}$$

Now, $49m^4 + 85m^3 + 72m^2 - 572m - 896 =$

$$\frac{49(729m^3 + 2754m^2 + 6984m + 5948)(9m - 19)}{6561} +$$

$$\frac{(2430m^2 + 288m + 44)(9m - 19)}{6561} - 51 - \frac{5621}{6561} \overset{via\ (m)}{<} 0$$

$\Rightarrow m(-49m^4 - 85m^3 - 72m^2 + 572m + 896) > 0$ and so,

in order to prove ③, it suffices to prove :

$$m^2(-49m^4 - 85m^3 - 72m^2 + 572m + 896)^2 \overset{?}{\ge}$$

$$1728m^4(-m^4 - 18m^3 - m^2 + 48m + 80) \Leftrightarrow$$

$$7m^2(m+1)(m-2)^2 \begin{pmatrix} 343m^5 + 2219m^4 + 8944m^3 + 23644m^2 + \\ 36608m + 28672 \end{pmatrix} \overset{?}{\ge} 0$$

$\to$ true $\because m > 0 \Rightarrow$ ③ $\Rightarrow$ ② $\Rightarrow$ ① is true ∴ $\sqrt{g_a n_a} \le \sqrt{\frac{7R}{6r} - \frac{4}{3}} . h_a \ \forall \Delta ABC$,

" = " iff Δ ABC is equilateral (QED)

4146. In any Δ ABC following relationship holds :

$$|m_b - m_c| \le \sqrt{\frac{1}{2}(s^2 + 2Rr - 31r^2)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$

$c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$ and then, we have the following set $''S''$ of relations : $y^2+z^2=x^2(m^2-2n)$, $y^3+z^3=x^3(m^3-3nn)$, $y^4+z^4=x^4((m^2-2n)^2-2n^2)$, $y^5+z^5=x^5\left(m((m^2-2n)^2+n^2-nm^2)\right)$, $y^6+z^6=x^6((m^3-3nn)^2-2n^3)$, and now,

$$|m_b-m_c|\overset{?}{\le}\sqrt{\frac{1}{2}(s^2+2Rr-31r^2)}\Leftrightarrow m_b^2+m_c^2-\frac{s^2+2Rr-31r^2}{2}\overset{?}{\le}2m_bm_c$$

and it's trivially true when : $m_b^2+m_b^2-\frac{s^2+2Rr-31r^2}{2}\le 0$ and now we focus on $m_b^2+m_b^2-\frac{s^2+21Rr-31r^2}{2}>0$ and then, it suffices to prove :

$$4m_b^2m_c^2\overset{?}{\ge}\left(m_b^2+m_c^2-\frac{s^2+2Rr-31r^2}{2}\right)^2$$

$$\Leftrightarrow\frac{1}{4}\left(\sum_{cyc}x\right)^2.\left(4y\left(\sum_{cyc}x\right)+(z-x)^2\right)\left(4z\left(\sum_{cyc}x\right)+(x-y)^2\right)\overset{?}{\ge}$$

$$\frac{1}{16}\left(\begin{array}{c}\left(\sum_{cyc}x\right)\left(4(y+z)\left(\sum_{cyc}x\right)+(z-x)^2+(x-y)^2\right)-\\ 2\left(\sum_{cyc}x\right)^2-\prod_{cyc}(y+z)+62xyz\end{array}\right)$$

$$\Leftrightarrow 4x^6+16x^5(y+z)-x^4(y^2+z^2)+134x^4yz-32x^3(y^3+z^3)+672x^3.yz(y+z)-10x^2(y^4+z^4)+366x^2.yz(y^2+z^2)-2844x^2y^2z^2+16x\left(y^5+z^5\right)-112x.yz(y^3+z^3)-48x.y^2z^2(y+z)+7(y^6+z^6)+76yz(y^4+z^4)+236y^2z^2(y^2+z^2)+334y^3z^3\boxed{\overset{?}{\underset{(1)}{\ge}}}0$$

and via set of relations $''S''$, in order to prove (1), it suffices to prove, following simplification :

$$\overbrace{(5m-58)(m-62)}^{\Omega_1}n^2-\overbrace{(34m^4-192m^3+406m^2+768m+136)}^{\Omega_2}n-\overbrace{(m-1)^2(7m^4+30m^3+43m^2+24m+4)}^{\Omega_3}\boxed{\overset{?}{\underset{(2)}{\le}}}0$$

Now, $\Omega_2=34m^4-192m^3+406m^2+768m+136=$

$$\frac{(1088m^2-7504m+12939)(4m^2+5m)+37732m^2+33609m+17408}{128}>0$$

($\because$ discriminant of $(1088m^2-7504m+12939)=-512<0$) $\therefore$ if $\frac{58}{5}\le m\le 62$, then : (2) is trivially true and so, we now consider the cases when :

$\boxed{m \in \left(0, \frac{58}{5}\right) \text{ or } m \in (62, \infty)}$ and now, LHS of ② is a quadratic polynomial in "n" with

$$\text{discriminant}, \boxed{\delta} = \Omega_2^2 + 4\Omega_1\Omega_3$$

$$= 144(m+1)^2\left(\begin{matrix}((3m^2-31)^2 + 26m^3 + 39m^2 + 656m + 480)(m-1)^2 + \\ 2256m + 48\end{matrix}\right) > 0,$$

hence, since $n \overset{AM-GM}{\leq} \frac{m^2}{4}$ ∴ in order to prove ②, it suffices to prove :

$$\Omega_1 . \frac{m^2}{2} \overset{?}{\leq} \Omega_2 + \sqrt{\delta} \Leftrightarrow \Omega_1 . m^2 - 2\Omega_2 \boxed{\underset{(m)}{\overset{?}{\leq}}} 2\sqrt{\delta} \text{ AND } \Omega_1 . 2n \boxed{\underset{(n)}{\overset{?}{\geq}}} \Omega_2 - \sqrt{\delta}$$

We note that (m) is trivially true if : $\Omega_1 . m^2 - 2\Omega_2 \leq 0$ and so, we now focus on the case when : $\Omega_1 . m^2 - 2\Omega_2 > 0$ and then :

$$63m^4 - 16m^3 - 2784m^2 + 1536m + 272 < 0$$

$$\Rightarrow (m-7)(63m^3 + 425m^2 + 191m + 2873) + 20383 < 0$$

$$\Rightarrow (m-7)(63m^3 + 425m^2 + 191m + 2873) < -20383 < 0 \Rightarrow \boxed{m < 7} \to \text{(i)}$$

and now, $\Omega_1 . m^2 - 2\Omega_2 > 0 \Rightarrow$ in order to prove (m), it suffices to prove :

$$\left(-(63m^4 - 16m^3 - 2784m^2 + 1536m + 272)\right)^2 \overset{?}{\leq}$$

$$576(m+1)^2\left(\left(\begin{matrix}(3m^2-31)^2 + 26m^3 + 39m^2 + \\ 656m + 480\end{matrix}\right)(m-1)^2 + 2256m + 48\right)$$

$$\Leftrightarrow (m-2)^2((m-7)-55)\left(\begin{matrix}1215m^4(m-7) - 1449m^4 - 46984m^3 - \\ 11664m^2 - 4560m - 928\end{matrix}\right) \overset{?}{\geq} 0$$

$\to$ true via (i) $\Rightarrow \boxed{\text{(m) is true}}$; Again, $\Omega_1 . 2n + \sqrt{\delta} > \sqrt{\delta}\ (\because > 0) = \sqrt{\Omega_2^2 + 4\Omega_1\Omega_3}$

$> \sqrt{\Omega_2^2}\ (\because \Omega_3 > 0) = \Omega_2\ (\because \Omega_2 > 0$, deduced earlier$) \Rightarrow \boxed{\text{(n) is true}}$ ∴ both

(m) and (n) are true ⇒ ② ⇒ ① is true ∴ $|m_b - m_c| \leq \sqrt{\frac{1}{2}(s^2 + 2Rr - 31r^2)}$

$\forall\, \Delta\, ABC$, " = " iff $\Delta\, ABC$ is equilateral (QED)

4147. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \cos(A)\right)\left(\sum_{cyc} sec^2(A)\right) \leq \frac{3R}{r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\left(\sum_{cyc} \cos(A)\right)\left(\sum_{cyc} sec^2(A)\right) = \left(\sum_{cyc} \cos(A)\right)\left(\sum_{cyc} \frac{1}{cos^2\left(\frac{A}{2}\right)}\right) =$$

$$=\left(1+\frac{r}{R}\right)\left(1+\frac{(4R+r)^2}{p^2}\right)\le\left(1+\frac{r}{R}\right)\left(1+\frac{(4R+r)^2}{\frac{27Rr}{2}}\right)=$$

$$=\left(1+\frac{r}{R}\right)\left(1+\frac{32R^2+16Rr+2r^2}{27Rr}\right)=\left(1+\frac{r}{R}\right)\left(\frac{32\left(\frac{R}{r}\right)^2+43\left(\frac{R}{r}\right)+2}{27.\left(\frac{R}{r}\right)}\right)$$

$$Let\ \frac{R}{r}=x\ge 2\ Then\ let's\ \ prove\ \ it$$

$$\left(1+\frac{1}{x}\right)\left(\frac{32x^2+43x+2}{27x}\right)\le 3x$$

$$\frac{x+1}{x}.\frac{32x^2+43x+2}{27x}\le 3x$$

$$49x^3-75x^2-45x-2\ge 0$$

$$(x-2)(49x^2+23x+1)\ge 0$$

$$x-2\ge 0\to\ \ x\ge 2\ \ (Euler)\ (Proved)$$

$$Equality\ \ holds\ \ for:\ \ \ a=b=c$$

4148. In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{m_a}{h_a}}.\sqrt{r_a}\le 3\sqrt{\frac{R}{2r}\left(\frac{R^2-Rr+r^2}{r}\right)}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\sqrt{\frac{m_a}{h_a}}.\sqrt{r_a}\overset{CSB}{\le}\sqrt{\sum_{cyc}\frac{m_a}{h_a}}.\sqrt{\sum_{cyc}r_a}\overset{Panaitopol}{\le}\sqrt{3\frac{R}{2r}}.\sqrt{(4R+r)}=$$

$$=3\sqrt{\frac{R}{2r}.\frac{4R+r}{3}}\le 3\sqrt{\frac{R}{2r}\left(\frac{R^2-Rr+r^2}{r}\right)}$$

$$Let's\ prove\ \ that:\ \frac{4R+r}{3}\le\frac{R^2-Rr+r^2}{r}$$

$$3R^2-7Rr+2r^2\ge 0$$

$$(R-2r)(3R-r)\ge 0\ \to\ \ R\ge 2r\ \ (Euler)$$

$$Equality\ \ holds\ \ for:\ \ \ a=b=c$$

4149. In ΔABC the following relationship holds:

$$108r^3 \le \sum_{cyc} a^2 h_a \le 27R^2 r$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\therefore \sum_{cyc} a^2 h_a = \sum_{cyc} a^2 . \frac{2F}{a} = 2F \sum_{cyc} a = 4pF = 4p^2 r \overset{Mitrinovici}{\lesseqgtr} 27R^2 r$$

$$\therefore \sum_{cyc} a^2 h_a = 4p^2 r \overset{Mitrinovici}{\gtreqless} 4r.27r^2 = 108r^3$$

$$Equality\ \ holds\ \ for: \ \ a = b = c$$

4150. In ΔABC the following relationship holds:

$$\frac{3}{4r} \le \sum_{cyc} \frac{r_a}{bc} \le \frac{1}{r} - \frac{1}{2R}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{r_a}{bc} = \frac{p.\tan\left(\frac{A}{2}\right)}{\frac{2F}{\sin(A)}} = \frac{p.\tan\left(\frac{A}{2}\right).\sin(A)}{2pr} = \frac{\sin\left(\frac{A}{2}\right).\sin(A)}{2r.\cos\left(\frac{A}{2}\right)} =$$

$$= \frac{2\sin^2\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)}{2r.\cos\left(\frac{A}{2}\right)} = \frac{\sin^2\left(\frac{A}{2}\right)}{r}$$

$$\sum_{cyc} \frac{\sin^2\left(\frac{A}{2}\right)}{r} = \frac{1}{r}\left(\sum_{cyc} \sin^2\left(\frac{A}{2}\right)\right) = \frac{1}{r}\left(1 - \frac{r}{2R}\right) = \frac{1}{r} - \frac{r}{2R} \ \ (RHS)$$

$$\sum_{cyc} \frac{\sin^2\left(\frac{A}{2}\right)}{r} = \frac{1}{r}\left(1 - \frac{r}{2R}\right) \ge \frac{1}{r}\left(1 - \frac{1}{4}\right) = \frac{3}{4r} \ (LHS)$$

$$Equality\ \ holds\ \ for: \ \ a = b = c$$

4151. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{h_b h_c}{bc} \le \sum_{cyc} \frac{r_b r_c}{bc}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$RHS=\sum_{cyc}\frac{r_br_a}{bc}=4F^2\sum_{cyc}\frac{1}{bc(p-b)(p-c)}$$
$$LHS=\sum_{cyc}\frac{h_bh_c}{bc}=4F^2\sum_{cyc}\frac{1}{(bc)^2}$$
$$Let's\ prove\ that\ RHS-LHS\geq 0$$
$$4F^2\left(\sum_{cyc}\frac{1}{bc(p-b)(p-c)}-\frac{1}{(bc)^2}\right)=$$
$$=4F^2\left(\sum_{cyc}\frac{bc-(p-b)(p-c)}{b^2c^2(p-b)(p-c)}\right)=4F^2\sum_{cyc}\frac{bc-p^2+pb+pc-bc}{b^2c^2(p-b)(p-c)}=$$
$$=4F^2\sum_{cyc}\frac{p(b+c)-p^2}{b^2c^2(p-b)(p-c)}=4F^2\sum_{cyc}\frac{p(2p-a)-p^2}{b^2c^2(p-b)(p-c)}=$$
$$=4F^2\sum_{cyc}\frac{p(p-a)}{b^2c^2(p-b)(p-c)}\geq 0$$
$$Equality\ holds\ for:\quad a=b=c$$

4152. In ΔABC the following relationship holds:

$$\sum_{cyc}h_a^2\cdot\left(\frac{1}{h_b^3}+\frac{1}{h_c^3}\right)\geq\frac{2}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}h_a^2\cdot\left(\frac{1}{h_b^3}+\frac{1}{h_c^3}\right)=\sum_{cyc}\left(\frac{h_a^2}{h_b^3}+\frac{h_a^2}{h_c^3}\right)=\sum_{cyc}\left(\frac{\frac{4F^2}{a^2}}{\frac{8F^3}{b^3}}+\frac{\frac{4F^2}{a^2}}{\frac{8F^3}{c^3}}\right)=$$
$$=\frac{1}{2F}\sum_{cyc}\left(\frac{b^3}{a^2}+\frac{c^3}{a^2}\right)\overset{Radon}{\geq}\frac{1}{2F}\cdot\frac{\left(2(a+b+c)\right)^3}{\left(2(a+b+c)\right)^2}=\frac{1}{2F}\cdot 2(a+b+c)=$$
$$=\frac{a+b+c}{F}=\frac{2p}{pr}=\frac{2}{r}$$
$$Equality\ holds\ for:\quad a=b=c$$

4153. In ΔABC the following relationship holds:

$$\frac{9r}{2R} \le \sum_{cyc} \frac{h_b . h_c}{bc} \le \frac{9}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \frac{h_b . h_c}{bc} = \sum_{cyc} \frac{\frac{4F^2}{bc}}{bc} = 4F^2 \sum_{cyc} \left(\frac{1}{bc}\right)^2 \overset{Bergstrom}{\ge}$$

$$\ge 4F^2 . \frac{\left(\sum_{cyc} \frac{1}{bc}\right)^2}{3} = 4F^2 . \frac{\left(\frac{a+b+c}{abc}\right)^2}{3} = 4F^2 . \frac{4p^2}{3(abc)^2} =$$

$$= \frac{16p^2F^2}{3.16F^2R^2} = \frac{p^2}{3R^2} \overset{Mitrinovici}{\ge} \frac{27Rr}{2.3R^2} = \frac{9r}{2R} \ (LHS)$$

$$\sum_{cyc} \frac{h_b . h_c}{bc} = 4F^2 \sum_{cyc} \left(\frac{1}{bc}\right)^2 = 4F^2 . \frac{a^2+b^2+c^2}{a^2.b^2.c^2} \overset{Leibniz}{\le} 4F^2 . \frac{9R^2}{16F^2R^2} = \frac{9}{4} \ (RHS)$$

$$Equality \ holds \ for: \ a = b = c$$

4154. In ΔABC the following relationship holds:

$$27r \le \sum_{cyc} r_a . ctg^2\left(\frac{A}{2}\right) \le \frac{27R^2}{4r}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} r_a . ctg^2\left(\frac{A}{2}\right) = \sum_{cyc} r_a . \left(\frac{p}{r_a}\right)^2 = p^2 \sum_{cyc} \frac{1}{r_a} = \frac{p^2}{r} \overset{Mitrinovici}{\le} \frac{27R^2}{4r} \ (RHS)$$

$$\sum_{cyc} r_a . ctg^2\left(\frac{A}{2}\right) = \frac{p^2}{r} \overset{Mitrinovici}{\ge} \frac{27r^2}{r} = 27r \ (LHS)$$

$$Equality \ holds \ for: \ a = b = c$$

4155. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{h_a}{bc} \le \sum_{cyc} \frac{r_a}{bc}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$LHS=\sum_{cyc}\frac{h_a}{bc}=\sum_{cyc}\frac{\frac{2F}{a}}{bc}=\sum_{cyc}\frac{2F}{abc}=\frac{6F}{abc}=\frac{6F}{4RF}=\frac{3}{2R}$$

$$RHS=\frac{r_a}{bc}+\frac{r_b}{ac}+\frac{r_c}{ba}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{r_ar_br_c}{(abc)^2}}=3\sqrt[3]{\frac{p^2r}{16F^2R^2}}=$$

$$=3\sqrt[3]{\frac{p^2r}{16p^2R^2r^2}}=3\sqrt[3]{\frac{1}{16rR^2}}\overset{Euler}{\geq}3\sqrt[3]{\frac{1}{8R^3}}=\frac{3}{2R}=LHS$$

$$So,\quad RHS\geq LHS\ \ (Proved)$$

$$Equality\ holds\ if:\ \ a=b=c$$

4156. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\tan^2\left(\frac{A}{2}\right)}{h_a}\geq\sum_{cyc}\frac{\tan^2\left(\frac{A}{2}\right)}{r_a}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$In\ \ \Delta ABC\ wlog\ a\leq b\leq c;\ h_a\geq h_b\geq h_c;\ r_a\leq r_b\leq r_c$$

$$\tan\left(\frac{A}{2}\right)\leq\tan\left(\frac{B}{2}\right)\leq\tan\left(\frac{C}{2}\right);\ \frac{1}{h_a}\leq\frac{1}{h_b}\leq\frac{1}{h_c};\ \frac{1}{r_a}\geq\frac{1}{r_b}\geq\frac{1}{r_c}$$

$$\sum_{cyc}\frac{\tan^2\left(\frac{A}{2}\right)}{h_a}\overset{Chebyshev}{\geq}\frac{1}{3}\sum_{cyc}\frac{1}{h_a}\cdot\sum_{cyc}\tan^2\left(\frac{A}{2}\right)=\frac{1}{3r}\sum_{cyc}\tan^2\left(\frac{A}{2}\right)\ (1)$$

$$\sum_{cyc}\frac{\tan^2\left(\frac{A}{2}\right)}{r_a}\overset{Chebyshev}{\leq}\frac{1}{3}\sum_{cyc}\frac{1}{r_a}\cdot\sum_{cyc}\tan^2\left(\frac{A}{2}\right)=\frac{1}{3r}\sum_{cyc}\tan^2\left(\frac{A}{2}\right)\ (2)$$

$$From\ \ (1)\ and\ (2)\ \ we\ \ obtain..$$

$$LHS\geq\frac{1}{3r}\sum_{cyc}\tan^2\left(\frac{A}{2}\right)\geq RHS\ (Proved)$$

$$Equality\ holds\ iff:\ \ a=b=c$$

4157. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a^2+b^2}{\sqrt{r_c}}\le 2\sum_{cyc}\frac{a^2}{\sqrt{h_a}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let }\sqrt{s-a}=x,\sqrt{s-b}=y,\sqrt{s-c}=z\ (\because (s-a),(s-b),(s-c)>0)$$

$$\text{and then}: s-a=x^2, s-b=y^2, s-c=z^2\Rightarrow 3s-2s=\sum_{cyc}x^2\text{ and so,}$$

$$a=y^2+z^2, b=z^2+x^2, c=x^2+y^2\ \therefore \sum_{cyc}\frac{b^2+c^2}{\sqrt{r_a}}\overset{?}{\le}2\sum_{cyc}\frac{a^2}{\sqrt{h_a}}\Leftrightarrow$$

$$\sum_{cyc}\frac{x((z^2+x^2)^2+(x^2+y^2)^2)}{\sqrt{rs}}\overset{?}{\le}\sum_{cyc}\frac{(y^2+z^2)^2.\sqrt{2(y^2+z^2)}}{\sqrt{rs}}$$

$$\Leftrightarrow\sum_{cyc}\left((y^2+z^2)^2.\sqrt{2(y^2+z^2)}\right)\boxed{\overset{?}{\underset{(*)}{\ge}}}\sum_{cyc}\left(x((z^2+x^2)^2+(x^2+y^2)^2)\right)$$

$$\text{Now,}\sum_{cyc}\left((y^2+z^2)^2.\sqrt{2(y^2+z^2)}\right)\ge\sum_{cyc}\left((y^2+z^2)^2(y+z)\right)$$

$$=(y^2+z^2)^2(y+z)+(z^2+x^2)^2(z+x)+(x^2+y^2)^2(x+y)$$

$$=x((z^2+x^2)^2+(x^2+y^2)^2)+y((x^2+y^2)^2+(y^2+z^2)^2)+$$

$$z((y^2+z^2)^2+(z^2+x^2)^2)=\sum_{cyc}\left(x((z^2+x^2)^2+(x^2+y^2)^2)\right)\Rightarrow(*)\text{ is true}$$

$$\text{and so,}\sum_{cyc}\frac{a^2+b^2}{\sqrt{r_c}}\le 2\sum_{cyc}\frac{a^2}{\sqrt{h_a}}\ \forall\ \Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Jenish Rijal-Nepal

$$\textit{Here, we know: }\frac{1}{\sqrt{h_a}}=\sqrt{\frac{1}{2}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)}\overset{\overbrace{\text{Power Mean}}}{\ge}\frac{1}{2}\left(\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\right)\Leftrightarrow\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\le\frac{2}{\sqrt{h_a}}$$

$$\textit{Now, we have: }\sum_{cyc}\frac{a^2+b^2}{\sqrt{r_c}}=\sum_{cyc}a^2\left(\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\right)\le\sum_{cyc}a^2\left(\frac{2}{\sqrt{h_a}}\right)=2\sum_{cyc}\frac{a^2}{\sqrt{h_a}}$$

Equality holds if and only if the triangle is equilateral.

4158. In acute ΔABC the following relationship holds:

$$\frac{HA}{a}+\frac{HB}{b}+\frac{HC}{c}\geq\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{HA}{a}+\frac{HB}{b}+\frac{HC}{c}=\sum_{cyc}\frac{HA}{a}=\sum_{cyc}\frac{2R\cos A}{2R\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{4F}=$$

$$=\frac{1}{4F}\left(\sum_{cyc}b^2+\sum_{cyc}c^2-\sum_{cyc}a^2\right)=\frac{1}{4F}\sum_{cyc}a^2=$$

$$=\frac{1}{4F}(a^2+b^2+c^2)\overset{IONESCU-WEITZENBOCK}{\geq}\frac{1}{4F}\cdot 4\sqrt{3}F=\sqrt{3}$$

Equality holds for $a=b=c$.

4159. In acute ΔABC the following relationship holds:

$$\frac{a}{HA\sin A}+\frac{b}{HB\sin B}+\frac{c}{HC\sin C}\geq 6$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{HA\sin A}+\frac{b}{HB\sin B}+\frac{c}{HC\sin C}=\sum_{cyc}\frac{a}{HA\sin A}=$$

$$=\sum_{cyc}\frac{2R\sin A}{2R\cos A\cdot\sin A}=\sum_{cyc}\frac{1}{\cos A}\overset{JENSEN}{\geq}3\cdot\frac{1}{\cos\left(\frac{A+B+C}{3}\right)}=\frac{3}{\cos\frac{\pi}{3}}=\frac{3}{\frac{1}{2}}=6$$

Equality holds for $a=b=c$.

4160. In ΔABC the following relationship holds:

$$r_ar_br_c\geq 27r^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$r_a r_b r_c = \frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c} = \frac{F^3}{(s-a)(s-b)(s-c)} =$$

$$= \frac{F^3 s}{s(s-a)(s-b)(s-c)} = \frac{F^3 s}{F^2} = Fs = rs\cdot s = rs^2 \overset{MITRINOVIC}{\geq}$$

$$\geq r\cdot\left(3\sqrt{3}r\right)^2 = r\cdot 27r^2 = 27r^3$$

Equality holds for $a=b=c$.

4161. In acute ΔABC the following relationship holds:

$$\frac{HA\sin A}{a} + \frac{HB\sin B}{b} + \frac{HC\sin C}{c} \leq \frac{3}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{HA\sin A}{a} + \frac{HB\sin B}{b} + \frac{HC\sin C}{c} = \sum_{cyc}\frac{HA\sin A}{a} = \sum_{cyc}\frac{2R\cos A\sin A}{2R\sin A} =$$

$$= \sum_{cyc}\cos A \overset{JENSEN}{\leq} 3\cos\left(\frac{A+B+C}{3}\right) = 3\cos\frac{\pi}{3} = \frac{3}{2}$$

Equality holds for $a=b=c$.

4162. In ΔABC the following relationship holds:

$$\frac{GA^2}{a^2} + \frac{GB^2}{b^2} + \frac{GC^2}{c^2} \geq \frac{3}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{GA^2}{a^2} + \frac{GB^2}{b^2} + \frac{GC^2}{c^2} = \sum_{cyc}\frac{GA^2}{a^2} = \frac{2}{3}\sum_{cyc}\frac{m_a^2}{a^2} =$$

$$= \frac{2}{3}\sum_{cyc}\frac{2(b^2+c^2)-a^2}{4a^2} = \frac{1}{6}\sum_{cyc}\left(\frac{2b^2}{a^2}+\frac{2c^2}{a^2}\right) - \frac{1}{6}\sum_{cyc}\frac{a^2}{a^2} =$$

$$=\frac{2}{6}\left(\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)+\left(\frac{c^2}{b^2}+\frac{b^2}{c^2}\right)+\left(\frac{a^2}{c^2}+\frac{c^2}{a^2}\right)\right)-\frac{1}{6}\cdot 3\geq$$

$$\geq\frac{1}{3}(2+2+2)-\frac{1}{2}=\frac{6}{3}-\frac{1}{2}=2-\frac{1}{2}=\frac{3}{2}$$

Equality holds for $a=b=c$.

4163. **In acute ΔABC the following relationship holds:**

$$\frac{a\cos A}{HA}+\frac{b\cos B}{HB}+\frac{c\cos C}{HC}\leq\frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a\cos A}{HA}+\frac{b\cos B}{HB}+\frac{c\cos C}{HC}=\sum_{cyc}\frac{a\cos A}{HA}=\sum_{cyc}\frac{2R\sin A\cdot\cos A}{2R\cos A}=\sum_{cyc}\sin A\overset{JENSEN}{\leq}$$

$$\leq 3\sin\left(\frac{A+B+C}{3}\right)=3\sin\frac{\pi}{3}=3\cdot\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$$

Equality holds for $a=b=c$.

4164. **In ΔABC the following relationship holds:**

$$GA^2+GB^2+GC^2\leq 3R^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$GA^2+GB^2+GC^2=\sum_{cyc}GA^2=\sum_{cyc}\left(\frac{2}{3}m_a\right)^2=\frac{4}{9}\sum_{cyc}m_a^2=$$

$$=\frac{4}{9}\cdot\frac{3}{4}(a^2+b^2+c^2)=\frac{1}{3}(a^2+b^2+c^2)=\frac{1}{3}\cdot 2(s^2-r^2-4Rr)\overset{GERRETSEN}{\leq}$$

$$\leq\frac{2}{3}(4R^2+3r^2+4Rr-r^2-4Rr)=\frac{2}{3}(4R^2+2r^2)=\frac{2}{3}\left(4R^2+2\cdot\frac{R^2}{4}\right)$$

$$=\frac{2}{3}\cdot\frac{18R^2}{4}=\frac{18R^2}{6}=3R^2$$

Equality holds for $a=b=c$.

4165. In ΔABC the following relationship holds:

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\ge\frac{6r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}=\sum_{cyc}\frac{h_a}{r_a}=\sum_{cyc}\frac{\frac{2F}{2}}{\frac{F}{s-a}}=2\sum_{cyc}\frac{s-a}{a}=2\left(s\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a}{a}\right)=$$

$$=2\left(s\cdot\frac{ab+bc+ca}{abc}-3\right)=2\left(s\cdot\frac{s^2+r^2+4Rr}{4Rrs}-3\right)=$$

$$=2\left(\frac{s^2+r^2+4Rr-12Rr}{4Rr}\right)\overset{GERRETSEN}{\ge}$$

$$\ge 2\cdot\frac{16Rr-5r^2+r^2-8Rr}{4Rr}=\frac{8Rr-4r^2}{2Rr}=\frac{4R-2r}{R}\overset{EULER}{\ge}\frac{4\cdot 2r-2r}{R}=\frac{6r}{R}$$

Equality holds for $a=b=c$.

4166. In ΔABC the following relationship holds:

$$\frac{h_a}{r_b}+\frac{h_b}{r_c}+\frac{h_c}{r_a}\ge\frac{6r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a}{r_b}+\frac{h_b}{r_c}+\frac{h_c}{r_a}=\sum_{cyc}\frac{\frac{2F}{a}}{\frac{F}{s-b}}=2\sum_{cyc}\frac{s-b}{a}\ge$$

$$\overset{AM-GM}{\ge}2\cdot 3\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{abc}}=6\sqrt[3]{\frac{s(s-a)(s-b)(s-c)}{s\cdot 4RF}}=$$

$$=6\sqrt[3]{\frac{F^2}{4sRF}}=6\sqrt[3]{\frac{F}{4sR}}=6\sqrt[3]{\frac{rs}{4sR}}=6\sqrt[3]{\frac{r}{4R}}\ge\frac{6r}{R}\Leftrightarrow$$

$$\Leftrightarrow\frac{r}{4R}\ge\left(\frac{r}{R}\right)^3\Leftrightarrow\frac{1}{4}\ge\frac{r^2}{R^2}\Leftrightarrow R^2\ge 4r^2\Leftrightarrow R\ge 2r$$

Equality holds for $a=b=c$.

4167. In ΔABC the following relationship holds:

$$\frac{h_a h_b}{r_a r_b}+\frac{h_b h_c}{r_b r_c}+\frac{h_c h_a}{r_c r_a}\ge\frac{12r^2}{R^2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a h_b}{r_a r_b}+\frac{h_b h_c}{r_b r_c}+\frac{h_c h_a}{r_c r_a}\overset{AM-GM}{\ge}3\sqrt[3]{\left(\frac{h_a h_b h_c}{r_a r_b r_c}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}\right)^2}=3\sqrt[3]{\left(\frac{8(s-a)(s-b)(s-c)}{abc}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{8s(s-a)(s-b)(s-c)}{sabc}\right)^2}=3\sqrt[3]{\left(\frac{8F^2}{s\cdot 4RF}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{2F}{Rs}\right)^2}=3\sqrt[3]{\left(\frac{2rs}{Rs}\right)^2}=3\sqrt[3]{4\left(\frac{r}{R}\right)^2}\ge 12\left(\frac{r}{R}\right)^2\Leftrightarrow 27\cdot 4\left(\frac{r}{R}\right)^2\ge 12^3\cdot\left(\frac{r}{R}\right)^6$$

$$3^3\cdot 4\ge 3^3\cdot 4^3\cdot\left(\frac{r}{R}\right)^4,\ \frac{1}{16}\ge\left(\frac{r}{R}\right)^4\Leftrightarrow\frac{1}{2}\ge\frac{r}{R}\Leftrightarrow R\ge 2r\quad\text{(Euler)}$$

Equality holds for $a=b=c$.

4168. In ΔABC the following relationship holds:

$$\frac{h_a h_b}{r_c^2}+\frac{h_b h_c}{r_a^2}+\frac{h_c h_a}{r_b^2}\ge 12\frac{r^2}{R^2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a h_b}{r_c^2}+\frac{h_b h_c}{r_a^2}+\frac{h_c h_a}{r_b^2}\ge 3\sqrt[3]{\left(\frac{h_a h_b h_c}{r_a r_b r_c}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}\right)^2}=3\sqrt[3]{\left(\frac{8(s-a)(s-b)(s-c)}{abc}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{8s(s-a)(s-b)(s-c)}{sabc}\right)^2}=3\sqrt[3]{\left(\frac{8F^2}{s\cdot 4RF}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{2F}{Rs}\right)^2}=3\sqrt[3]{\left(\frac{2rs}{RS}\right)^2}=3\sqrt[3]{4\left(\frac{r}{R}\right)^2}\geq 12\left(\frac{r}{R}\right)^2\Leftrightarrow$$

$$27\cdot 4\left(\frac{r}{R}\right)^2\geq 12^3\cdot\left(\frac{r}{R}\right)^6\Leftrightarrow 3^3\cdot 4\geq 3^3\cdot 4^3\cdot\left(\frac{r}{R}\right)^4\Leftrightarrow\frac{1}{16}\geq\left(\frac{r}{R}\right)^4\Leftrightarrow\frac{1}{2}\geq\frac{r}{R}\Leftrightarrow R\geq 2r$$

Equality holds for $a=b=c$.

4169. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cot A}{\cot B}\geq\sum_{cyc}\frac{\sin^2 A}{\sin^2 B}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a\leq b\leq c\Rightarrow\begin{cases}b-a\geq b-c\geq a-c\\ \cot A\leq\cot B\leq\cot C\end{cases}$

$$\sum_{cyc}(b-a)\cot A\overset{CEBYSHEV}{\geq}\frac{1}{3}\sum_{cyc}(b-a)\cdot\sum_{cyc}\cot A=0\Leftrightarrow\sum_{cyc}(b-a)\cot A\geq 0$$

$$\sum_{cyc}(b+a)(b-a)\cot A\geq 0\Rightarrow\sum_{cyc}(b^2-a^2)\cot A\geq 0$$

$$\Rightarrow\sum_{cyc}(b^2-a^2)\cdot\frac{b^2+c^2-a^2}{4F}\geq 0\Rightarrow\sum_{cyc}(b^2-a^2)(b^2+c^2-a^2)\geq 0$$

$$\sum_{cyc}(b^4+b^2c^2-b^2a^2-a^2b^2-a^2c^2+a^4)\geq 0$$

$$\sum_{cyc}\left(a^2(b^2+c^2-a^2)-b^2(a^2+c^2-b^2)\right)\geq 0$$

$$\sum_{cyc}\frac{b^2+c^2-a^2}{a^2+c^2-b^2}\geq\sum_{cyc}\frac{b^2}{a^2}\Rightarrow\sum_{cyc}\frac{\cot A}{\cot B}\geq\sum_{cyc}\frac{\sin^2 A}{\sin^2 B}$$

Equality holds for $a=b=c$.

4170. In ΔABC the following relationship holds:

$$\frac{\sin A+\sin B}{\cos\frac{C}{2}}+\frac{\sin B+\sin C}{\cos\frac{A}{2}}+\frac{\sin C+\sin A}{\cos\frac{B}{2}}\leq 6$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\frac{\sin A+\sin B}{\cos\frac{C}{2}}=\sum_{cyc}\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\cos\frac{C}{2}}=$$

$$=2\sum_{cyc}\frac{\sin\frac{\pi-C}{2}\cos\frac{A-B}{2}}{\cos\frac{C}{2}}=2\sum_{cyc}\frac{\cos\frac{C}{2}\cos\frac{A-B}{2}}{\cos\frac{C}{2}}=2\sum_{cyc}\cos\frac{A-B}{2}\overset{JENSEN}{\leq}$$

$$\leq 2\cdot 3\cdot\cos\left(\frac{\frac{A-B}{2}+\frac{B-C}{2}+\frac{C-A}{2}}{3}\right)=6\cos\left(\frac{0}{3}\right)=6\cos 0=6\cdot 1=6$$

Equality holds for $A=B=C$.

4171. In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c^3}+\frac{b^4+c^4}{h_a^3}+\frac{c^4+a^4}{h_b^3}\geq 32r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a^4+b^4}{h_c^3}+\frac{b^4+c^4}{h_a^3}+\frac{c^4+a^4}{h_b^3}=\sum_{cyc}\frac{a^4+b^4}{h_c^3}\geq$$

$$\overset{AM-GM}{\geq}2\sum_{cyc}\frac{a^2b^2}{h_c^3}\overset{AM-GM}{\geq}2\cdot 3\sqrt[3]{\frac{(abc)^4}{(h_ah_bh_c)^3}}=\frac{6abc}{h_ah_bh_c}\sqrt[3]{abc}=$$

$$=\frac{6abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}\sqrt[3]{4RF}=\frac{6(abc)^2}{8F^3}\sqrt[3]{4Rrs}\geq$$

$$\overset{EULER}{\geq}\frac{6\cdot 16R^2F^2}{8F^3}\sqrt[3]{8r^2s}\overset{MITRINOVIC}{\geq}\frac{12R^2}{F}\sqrt[3]{8r^2\cdot 3\sqrt{3}r}=$$

$$=\frac{12R^2}{rs}\cdot 2\sqrt{3}r=\frac{24\sqrt{3}R^2}{s}\overset{MITRINOVIC}{\geq}\frac{24\sqrt{3}\cdot R^2}{\frac{3\sqrt{3}}{2}R}=16R\overset{EULER}{\geq}16\cdot 2r=32r$$

Equality holds for $a=b=c$.

4172. In ΔABC the following relationship holds:

$$\sqrt{a}\sin A+\sqrt{b}\sin B+\sqrt{c}\sin C\leq\frac{3}{2}\sqrt[4]{27R^2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sqrt{a}\sin A+\sqrt{b}\sin B+\sqrt{c}\sin C\overset{CBS}{\leq}\sqrt{(a+b+c)(\sin^2 A+\sin^2 B+\sin^2 C)}=$$

$$=\sqrt{2s\cdot\frac{s^2-r^2-4Rr}{2R^2}}\overset{GERRETSEN}{\leq}\sqrt{\frac{s}{R^2}(4R^2+4Rr+3r^2-r^2-4Rr)}\overset{MITRINOVIC}{\leq}$$

$$\leq\sqrt{\frac{3\sqrt{3}}{2}R\cdot\frac{1}{R^2}\cdot(4R^2+2r^2)}\overset{EULER}{\leq}\sqrt{\frac{3\sqrt{3}}{2R}\left(4R^2+2\cdot\frac{R^2}{4}\right)}=\sqrt{\frac{3\sqrt{3}}{2R}\cdot\frac{9R^2}{2}}=$$

$$=\frac{3}{2}\sqrt{3\sqrt{3}R}=\frac{3}{2}\sqrt[4]{27R^2}$$

Equality holds for $a=b=c$.

4173. In ΔABC the following relationship holds:

$$\sqrt{a\sin^3 A}+\sqrt{b\sin^3 B}+\sqrt{c\sin^3 C}\leq\frac{9\sqrt{2R}}{4}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sqrt{a\sin^3 A}+\sqrt{b\sin^3 B}+\sqrt{c\sin^3 C}=\sum_{cyc}\sqrt{a\sin^3 A}=$$

$$=\sum_{cyc}\sqrt{2R\sin^4 A}=\sqrt{2R}\sum_{cyc}\sin^2 A=\sqrt{2R}\sum_{cyc}\frac{a^2}{4R^2}=$$

$$=\frac{\sqrt{2R}}{4R^2}\cdot\sum_{cyc}a^2=\frac{\sqrt{2R}}{4R^2}\cdot 2(s^2-r^2-4Rr)\le$$

$$\overset{GERRETSEN}{\le}\frac{\sqrt{2R}}{2R^2}(4R^2+4Rr+3r^2-r^2-4Rr)=$$

$$=\frac{\sqrt{2R}}{2R^2}(4R^2+2r^2)\overset{EULER}{\le}\frac{\sqrt{2R}}{2R^2}\left(4R^2+2\cdot\frac{R^2}{4}\right)=\frac{\sqrt{2R}}{2R^2}\cdot\frac{18R^2}{4}=\frac{9\sqrt{2R}}{4}$$

Equality holds for $a=b=c$.

4174. In ΔABC the following relationship holds:

$$\sqrt{a\cos\frac{A}{2}}+\sqrt{b\cos\frac{B}{2}}+\sqrt{c\cos\frac{C}{2}}\le\frac{3\sqrt{6R}}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sqrt{a\cos\frac{A}{2}}+\sqrt{b\cos\frac{B}{2}}+\sqrt{c\cos\frac{C}{2}}\overset{CBS}{\le}\sqrt{\sum_{cyc}a\cdot\sum_{cyc}\cos\frac{A}{2}}\overset{JENSEN}{\le}$$

$$\le\sqrt{2s\cdot 3\cos\left(\frac{A+B+C}{6}\right)}=\sqrt{6s\cdot\cos\frac{\pi}{6}}=\sqrt{6s\cdot\frac{\sqrt{3}}{2}}=\sqrt{3\sqrt{3}s}$$

Remains to prove:

$$\sqrt{3\sqrt{3}s}\le\frac{3\sqrt{6R}}{2}\Leftrightarrow 3\sqrt{3}s\le\frac{54R}{2}3\sqrt{3}s\le 27R\Leftrightarrow s\le\frac{9R}{\sqrt{3}}\Leftrightarrow s\le\frac{3\sqrt{3}}{2}R\quad\text{(MITRINOVIC)}$$

Equality holds for $a=b=c$.

4175. In ΔABC the following relationship holds:

$$\sqrt{a\sin\frac{A}{2}}+\sqrt{b\sin\frac{B}{2}}+\sqrt{c\sin\frac{C}{2}}\le\frac{3\sqrt[4]{12R^2}}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sqrt{a\sin\frac{A}{2}}+\sqrt{b\sin\frac{B}{2}}+\sqrt{c\sin\frac{C}{2}}\overset{CBS}{\leq}\sqrt{\sum_{cyc}a\cdot\sum_{cyc}\sin\frac{A}{2}}\overset{JENSEN}{\geq}\sqrt{2s\cdot 3\sin\left(\frac{A+B+C}{6}\right)}=$$

$$=\sqrt{6s\cdot\sin\frac{\pi}{6}}=\sqrt{6s\cdot\frac{1}{2}}=\sqrt{3s}$$

Remains to prove:

$$\sqrt{3s}\leq\frac{3\cdot\sqrt[4]{12R^2}}{2}\Leftrightarrow 3s\leq\frac{9\sqrt{12R^2}}{4}\Leftrightarrow 4s\leq 3\sqrt{12R^2}\Leftrightarrow s\leq\frac{3\cdot 2\sqrt{3}R}{4}\Leftrightarrow s\leq\frac{3\sqrt{3}R}{2}$$

(MITRINOVIC). Equality holds for $a=b=c$.

4176. In ΔABC the following relationship holds:

$$\sqrt{a\sin A}+\sqrt{b\sin B}+\sqrt{c\sin C}\leq\frac{3\sqrt{6R}}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sqrt{a\sin A}+\sqrt{b\sin B}+\sqrt{c\sin C}=\sum_{cyc}\sqrt{a\sin A}=$$

$$=\sum_{cyc}\sqrt{a\cdot\frac{a}{2R}}=\sum_{cyc}\frac{a}{\sqrt{2R}}=\frac{2s}{\sqrt{2R}}=\frac{s\sqrt{2}}{\sqrt{R}}$$

Remains to prove:

$$\frac{s\sqrt{2}}{\sqrt{R}}\leq\frac{3\sqrt{6R}}{2}\Leftrightarrow\frac{2s^2}{R}\leq\frac{9\cdot 6R}{4}$$

$$8s^2\leq 54R^2\Leftrightarrow s^2\leq\frac{27R^2}{4}\Leftrightarrow s\leq\frac{3\sqrt{3}R}{2}\quad\text{(MITRINOVIC)}$$

Equality holds for $a=b=c$.

4177. In ΔABC the following relationship holds:

$$\cos A\cos B\cos C\leq\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

We use:

$$\cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2},\qquad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}$$

We must prove:

$$\frac{s^2-(2R+r)^2}{4R^2}\le\frac{r}{4R},\qquad s^2-(2R+r)^2\le Rr$$

$$s^2\le 4R^2+4Rr+r^2+Rr=4R^2+5Rr+r^2$$

$$s^2\overset{GERRETSEN}{\le}4R^2+4Rr+3r^2\le 4R^2+5Rr+r^2\Leftrightarrow$$

$$\Leftrightarrow 3r^2\le Rr+r^2\Leftrightarrow Rr\ge 2r^2\Leftrightarrow R\ge 2r\ \text{(Euler)}$$

Equality holds for $a=b=c$.

4178. In ΔABC the following relationship holds:

$$\sin A\sin B\sin C\le\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

We use: $\sin A\sin B\sin C=\frac{sr}{2R^2};\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{4R}$

We must prove that:

$$\frac{sr}{2R^2}\le\frac{s}{4R}\Leftrightarrow\frac{r}{2R^2}\le\frac{1}{4R}\Leftrightarrow 2r\le R\quad\text{(Euler)}$$

Equality holds for $A=B=C$.

4179. In ΔABC the following relationship holds:

$$\frac{\sin A}{r_a}+\frac{\sin B}{r_b}+\frac{\sin C}{r_c}\le\frac{\sqrt{3}}{2r}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{\sin A}{r_a}+\frac{\sin B}{r_b}+\frac{\sin C}{r_c}=\sum_{cyc}\frac{\sin A}{r_a}=\sum_{cyc}\frac{\frac{a}{2R}}{\frac{F}{s-a}}=$$

$$=\frac{1}{2RF}\sum_{cyc}a(s-a)=\frac{1}{2RF}\left(s\sum_{cyc}a-\sum_{cyc}a^2\right)=$$

$$=\frac{1}{2RF}(s\cdot 2s-2s^2+2r^2+8Rr)=\frac{2r(r+4R)}{2RF}=$$

$$=\frac{r(r+4R)}{Rrs}=\frac{r+4R}{Rs}\overset{EULER}{\leq}\frac{\frac{R}{2}+4R}{Rs}=\frac{\frac{1}{2}+4}{s}\overset{MITRINOVIC}{\leq}\frac{\frac{9}{2}}{3\sqrt{3}r}=\frac{3}{2\sqrt{3}r}=\frac{\sqrt{3}}{2r}$$

Equality holds for $a=b=c$.

4180. In ΔABC the following relationship holds:

$$\frac{\sin A}{r_b}+\frac{\sin B}{r_c}+\frac{\sin C}{r_a}\geq\frac{\sqrt{3}}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{\sin A}{r_b}+\frac{\sin B}{r_c}+\frac{\sin C}{r_a}\geq 3\sqrt[3]{\frac{\sin A\sin B\sin C}{r_ar_br_c}}=$$

$$=3\sqrt[3]{\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}\cdot\frac{(s-a)(s-b)(s-c)}{F^3}}=\frac{3}{2R}\sqrt[3]{abc\cdot\frac{F^2}{sF^3}}=\frac{3}{2R}\sqrt[3]{\frac{abc}{sF}}=$$

$$=\frac{3}{2R}\cdot\sqrt[3]{\frac{4RF}{sF}}=\frac{3}{2R}\sqrt[3]{\frac{4R}{s}}\geq\frac{3}{2R}\sqrt[3]{4\cdot\frac{2}{3\sqrt{3}}\cdot\frac{s}{s}}=\frac{3}{2R}\cdot\frac{2}{\sqrt{3}}=\frac{\sqrt{3}}{R}$$

Equality holds for $a=b=c$

4181. In ΔABC the following relationship holds:

$$\frac{A}{a}+\frac{B}{b}+\frac{C}{c}\geq\frac{\pi\sqrt{3}}{3R}$$

Proposed by Pavlos Trifon-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

It is known that in ΔABC holds: $\frac{\sin A}{A}\cdot\frac{\sin B}{B}\cdot\frac{\sin C}{C}\leq\left(\frac{3\sqrt{3}}{2\pi}\right)^3$

$$\frac{A}{a}+\frac{B}{b}+\frac{C}{c}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{A\cdot B\cdot C}{a\cdot b\cdot c}}=3\sqrt[3]{\frac{1}{(2R)^3}\cdot\left(\frac{2\pi}{3\sqrt{3}}\right)^3}=3\cdot\frac{1}{2R}\cdot\frac{2\pi}{3\sqrt{3}}=\frac{\pi\sqrt{3}}{3R}$$

Equality holds if $a=b=c$

4182. In any Δ ABC the following relationship holds :

$$\frac{ah_b}{r_c}+\frac{bh_c}{r_a}+\frac{ch_a}{r_b}\geq\frac{12\sqrt{3}r^2}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}\overset{AM-GM}{\geq}3.\sqrt[3]{\frac{abc.h_ah_bh_c}{r_ar_br_c}}=3.\sqrt[3]{\frac{4Rrs.2r^2s^2}{Rrs^2}}=6.\sqrt[3]{r^2s}\overset{\text{Mitrinovic}}{\geq}\frac{6\sqrt{3}.r^2}{r}$$

$$\overset{\text{Euler}}{\geq}=12\sqrt{3}.\frac{r^2}{R}\text{ and so, }\frac{ah_b}{r_c}+\frac{bh_c}{r_a}+\frac{ch_a}{r_b}\geq12\sqrt{3}.\frac{r^2}{R}\ \forall\ \Delta\text{ ABC,}$$

$''=''$ iff Δ ABC is equilateral (QED)

4183. In any Δ ABC the following relationship holds :

$$\frac{a+h_b}{r_c}+\frac{b+h_c}{r_a}+\frac{c+h_a}{r_b}\geq\left(6+4\sqrt{3}\right).\frac{r}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}\overset{AM-GM}{\geq}3.\sqrt[3]{\frac{abc}{r_ar_br_c}}+3.\sqrt[3]{\frac{h_ah_bh_c}{r_ar_br_c}}=3.\sqrt[3]{\frac{4Rrs}{rs^2}}+3.\sqrt[3]{\frac{2r^2s^2}{Rrs^2}}\overset{\text{Mitrinovic}}{\geq}$$

$$3.\sqrt[3]{\frac{4}{s}.\frac{2s}{3\sqrt{3}}}+3.\sqrt[3]{\frac{8r^3}{R.4r^2}}\overset{\text{Euler}}{\geq}\frac{6r}{\sqrt{3}.r}+\frac{6r}{R}\overset{\text{Euler}}{\geq}\frac{12r}{\sqrt{3}.R}+\frac{6r}{R}=\left(6+4\sqrt{3}\right).\frac{r}{R}$$

$$\text{and so, }\frac{a+h_b}{r_c}+\frac{b+h_c}{r_a}+\frac{c+h_a}{r_b}\geq\left(6+4\sqrt{3}\right).\frac{r}{R}\ \forall\ \Delta\text{ ABC,}$$

$''=''$ iff Δ ABC is equilateral (QED)

4185. In ΔABC the following relationship holds:

$$\frac{r_a+r_b}{\sqrt{r_a^2+r_ar_b+r_b^2}}+\frac{r_c+r_b}{\sqrt{r_c^2+r_cr_b+r_b^2}}+\frac{r_a+r_c}{\sqrt{r_a^2+r_ar_c+r_c^2}}\ge\frac{4\sqrt{3}r}{\sqrt{3R^2-8r^2}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{r_a+r_b}{\sqrt{r_a^2+r_ar_b+r_b^2}}=\sum_{cyc}\frac{(r_a+r_b)^{\frac{3}{2}}}{\sqrt{(r_a^2+r_ar_b+r_b^2)(r_a+r_b)}}\overset{Radon}{\ge}$$

$$\ge\frac{\left(2(r_a+r_b+r_c)\right)^{\frac{3}{2}}}{\left(\sum_{cyc}(r_a^2+r_ar_b+r_b^2)(r_a+r_b)\right)^{\frac{1}{2}}}\ge\frac{\left(2(r_a+r_b+r_c)\right)^{\frac{3}{2}}}{\left(\frac{27}{4}R^2.2(r_a+r_b+r_c)\right)^{\frac{1}{2}}}=$$

$$=\frac{2\sqrt{2}}{\frac{3\sqrt{3}R}{\sqrt{2}}}(r_a+r_b+r_c)=\frac{4(4R+r)}{3\sqrt{3}R}\overset{Euler}{\ge}\frac{4.9r}{3\sqrt{3}R}=\frac{4\sqrt{3}r}{R}\ge\frac{4\sqrt{3}r}{\sqrt{3R^2-8r^2}}$$

$$\therefore R\le\sqrt{3R^2-8r^2}\to R^2\le 3R^2-8r^2\to R^2\ge 4r^2\to R\ge 2r\ (True)$$

$$Equality\ holds\ for:\ a=b=c$$

4185. In ΔABC the following relationship holds:

$$\frac{m_a+m_b}{\sqrt{m_a^2+m_am_b+m_b^2}}+\frac{m_c+m_b}{\sqrt{m_c^2+m_cm_b+m_b^2}}+\frac{m_a+m_c}{\sqrt{m_a^2+m_am_c+m_c^2}}\ge\frac{4\sqrt{3}r}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{m_a+m_b}{\sqrt{m_a^2+m_am_b+m_b^2}}=\sum_{cyc}\frac{(m_a+m_b)^{\frac{3}{2}}}{\sqrt{(m_a^2+m_am_b+m_b^2)(m_a+m_b)}}\overset{Radon}{\ge}$$

$$\frac{\left(2(m_a+m_b+m_c)\right)^{\frac{3}{2}}}{\left(\sum_{cyc}(m_a^2+m_am_b+m_b^2)(m_a+m_b)\right)^{\frac{1}{2}}}\ge\frac{\left(2(m_a+m_b+m_c)\right)^{\frac{3}{2}}}{\left(\frac{27}{4}R^2.2(m_a+m_b+m_c)\right)^{\frac{1}{2}}}=$$

$$=\frac{2\sqrt{2}}{\frac{3\sqrt{3}R}{\sqrt{2}}}(m_a+m_b+m_c)\overset{Gotman}{\ge}\frac{4.9r}{3\sqrt{3}R}=\frac{4\sqrt{3}r}{R}$$

$$\therefore m_a^2+m_am_b+m_b^2=\frac{3}{4}(a^2+b^2+c^2)\overset{Leibniz}{\leq}\frac{3}{4}.9R^2=\frac{27}{4}R^2$$

$$Equality\ \ holds\ \ for:\quad a=b=c$$

4186.

$$\left\{\begin{array}{c}\text{In any }\Delta\text{ ABC such that } a\neq b\neq c,\text{ the following relationship holds :}\\ \dfrac{a^2r_a}{(a-b)(a-c)}+\dfrac{b^2r_b}{(b-c)(b-a)}+\dfrac{c^2r_c}{(c-a)(c-b)}<\dfrac{27R^2}{4r}\\ \text{In any }\Delta\text{ ABC, the following relationship holds : } abc\geq 24\sqrt{3}.r^3\end{array}\right\}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2r_a}{(a-b)(a-c)}+\frac{b^2r_b}{(b-c)(b-a)}+\frac{c^2r_c}{(c-a)(c-b)}=$$

$$=\frac{-rs}{(a-b)(b-c)(c-a)r^2s}.\sum_{cyc}\Big(a^2(b-c)(s-b)(s-c)\Big)=$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}.\sum_{cyc}\Big(a^2(b-c)(-s^2+sa+bc)\Big)=$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}.\left(\begin{array}{c}-s^2\sum_{cyc}\Big(a^2(b-c)\Big)+s\sum_{cyc}\Big(a^3(b-c)\Big)+\\ abc\sum_{cyc}(a(b-c))\end{array}\right)$$

$$\therefore\frac{a^2r_a}{(a-b)(a-c)}+\frac{b^2r_b}{(b-c)(b-a)}+\frac{c^2r_c}{(c-a)(c-b)}=$$

$$\frac{-1}{(a-b)(b-c)(c-a)r}.\left(-s^2\sum_{cyc}\Big(a^2(b-c)\Big)+s\sum_{cyc}\Big(a^3(b-c)\Big)\right)\to(1)$$

$$\text{Now, }\sum_{cyc}\Big(a^2(b-c)\Big)=a^2(b-c)+b^2(c-a)+c^2(a-b)=$$

$$a^2(b-c)+bc(b-c)-a(b-c)(b+c)=(b-c)\Big((a^2-ab)-(ac-bc)\Big)$$

$$=-(b-c)(a-b)(c-a)\Rightarrow-s^2\sum_{cyc}\Big(a^2(b-c)\Big)=s^2(a-b)(b-c)(c-a)\to(2)$$

$$\text{Again, }\sum_{cyc}\Big(a^3(b-c)\Big)=a^3(b-c)+b^3(c-a)+c^3(a-b)$$

$$=a^3(b-c)+bc(b-c)(b+c)-a(b-c)(b^2+bc+c^2)$$

$$= (\mathbf{b}-\mathbf{c})(a^3+\mathbf{b}^2\mathbf{c}+\mathbf{b}\mathbf{c}^2-a\mathbf{b}^2-a\mathbf{b}\mathbf{c}-a\mathbf{c}^2)$$

$$= (\mathbf{b}-\mathbf{c})\left(a(a-\mathbf{b})(a+\mathbf{b})-\mathbf{c}^2(a-\mathbf{b})-\mathbf{b}\mathbf{c}(a-\mathbf{b})\right)$$

$$= (\mathbf{b}-\mathbf{c})(a-\mathbf{b})\left((a^2-\mathbf{c}^2)+(a\mathbf{b}-\mathbf{b}\mathbf{c})\right) = -(\mathbf{b}-\mathbf{c})(a-\mathbf{b})(\mathbf{c}-a)(a+\mathbf{b}+\mathbf{c})$$

$$\Rightarrow \mathbf{s}\sum_{\text{cyc}}\left(a^3(\mathbf{b}-\mathbf{c})\right) = -2\mathbf{s}^2(a-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-a) \to ③ \therefore ①,② \text{ and } ③ \Rightarrow$$

$$\frac{a^2\mathbf{r}_a}{(a-\mathbf{b})(a-\mathbf{c})}+\frac{\mathbf{b}^2\mathbf{r}_\mathbf{b}}{(\mathbf{b}-\mathbf{c})(\mathbf{b}-a)}+\frac{\mathbf{c}^2\mathbf{r}_\mathbf{c}}{(\mathbf{c}-a)(\mathbf{c}-\mathbf{b})} =$$

$$\frac{-1}{(a-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-a)\mathbf{r}}.\left(-\mathbf{s}^2(a-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-a)\right) = \frac{\mathbf{s}^2}{\mathbf{r}} \overset{\text{Mitrinovic}}{<} \frac{27\mathbf{R}^2}{4\mathbf{r}};$$

equality doesn't occur because $a \neq \mathbf{b} \neq \mathbf{c}$ and so,

$$\frac{a^2\mathbf{r}_a}{(a-\mathbf{b})(a-\mathbf{c})}+\frac{\mathbf{b}^2\mathbf{r}_\mathbf{b}}{(\mathbf{b}-\mathbf{c})(\mathbf{b}-a)}+\frac{\mathbf{c}^2\mathbf{r}_\mathbf{c}}{(\mathbf{c}-a)(\mathbf{c}-\mathbf{b})} < \frac{27\mathbf{R}^2}{4\mathbf{r}}\ \forall\,\Delta\,\mathbf{ABC}\ (\mathbf{QED})$$

$$\text{Also}, a\mathbf{b}\mathbf{c} = 4\mathbf{Rrs} \overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Euler}}}{\geq} 8\mathbf{r}^2.3\sqrt{3}.\mathbf{r} = 24\sqrt{3}.\mathbf{r}^3\ \forall\,\Delta\,\mathbf{ABC},$$

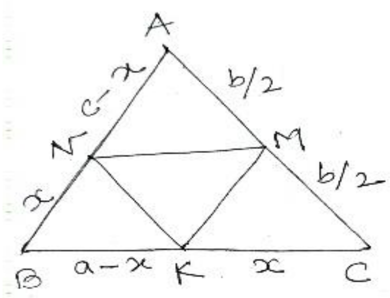
" = " iff Δ ABC is equilateral (QED)

4187. ***In ΔABC, let $M \in AC, K \in BC$, $N \in AB$ where M is the midpoint of AC, then:***

$$[MNK]_{\min.} = \frac{6abc - ba^2 - bc^2}{64R}$$

Proposed by Adgozel Adgozalov-Azerbaijan

Solution by Tapas Das-India



$$\text{Let } BN = KC = x\ \&\ 0 \leq x < \min(a,c),$$

$$[AMN] = \frac{1}{2}\left(\frac{b}{2}\right).(c-x)\sin A = \frac{c-x}{2c}[ABC] \text{as}$$

$$\left([ABC] = \frac{1}{2}bc\sin A\ \ \text{then} \sin A = \frac{2}{bc}[ABC]\right)$$

$$similarly, [CMK] = \frac{1}{2}\cdot\frac{b}{2}.xsinC = \frac{x}{2a}[ABC] \& [BNK] = \frac{1}{2}x(x-a)[ABC]$$

$$Now, [MNK] = [ABC] - [AMN] - [BNK] - [CMK] or, \frac{[MNK]}{[ABC]}$$

$$= 1 - \frac{c-x}{2c} - \frac{x(a-x)}{ac} - \frac{x}{2a} = \frac{1}{2ac}(2x^2 - (a+c)x + ac)\ (1)$$

$$let\ f(x) = 2x^2 - (a+c)x + ac\ , f'(x) = 4x - (a+c), f''(x) = 4 > 0\ , f'(x)$$

$$= 0\ gives\ x = \frac{a+c}{4}\ and\ f'\left(\frac{a+c}{4}\right) > 0$$

$$so\ f(x) minimum\ at\ x = \frac{a+c}{4}\ \&\ f\left(\frac{a+c}{4}\right) = \frac{2(a+c)^2}{16} - \frac{(a+c)^2}{4} + ac$$

$$= \frac{6ac - a^2 - c^2}{8}$$

$$From (1) we\ get$$

$$[MNK]_{\text{min.}} = \frac{6ac - a^2 - c^2}{8\times 2ac}.[ABC] = \frac{6ac - a^2 - c^2}{8\times 2ac}.\frac{abc}{4R} = \frac{6abc - ba^2 - bc^2}{64R}$$

4188. **In any acute ΔABC the following relationship holds:**

$$\frac{a\cot a}{h_a} + \frac{b\cot B}{h_b} + \frac{c\cot C}{h_c} = 2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a\cot A}{h_a} + \frac{b\cot B}{h_b} + \frac{c\cot C}{h_c} = \sum_{cyc}\frac{a\cot A}{h_a} = \sum_{cy}\frac{a}{\frac{2F}{a}}\cdot\frac{\cos A}{\sin A} = \frac{1}{2F}\sum_{cyc}a^2\cdot\frac{b^2+c^2-a^2}{2bc\sin A} =$$

$$= \frac{1}{2F}\cdot\frac{1}{4F}\sum_{cyc}a^2(b^2+c^2-a^2) =$$

$$= \frac{1}{8F^2}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4) = \frac{1}{8F^2}\cdot 16F^2 =$$

4189. **In ΔABC the following relationship holds:**

$$\frac{ab}{\sin\frac{C}{2}} + \frac{bc}{\sin\frac{A}{2}} + \frac{ca}{\sin\frac{B}{2}} \geq 72r^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{ab}{\sin\frac{C}{2}}+\frac{bc}{\sin\frac{A}{2}}+\frac{ca}{\sin\frac{B}{2}}=\sum_{cyc}\frac{ab}{\sin\frac{C}{2}}\overset{CEBYSHEV}{\ge}\frac{1}{3}\sum_{cyc}ab\cdot\sum_{cyc}\frac{1}{\sin\frac{C}{2}}\overset{JENSEN}{\ge}$$

$$\ge\frac{1}{3}(s^2+r^2+4Rr)\cdot\frac{3}{\sin\left(\frac{A+B+C}{6}\right)}\overset{MITRINOVIC}{\ge}(27r^2+r^2+4Rr)\cdot\frac{1}{\sin\frac{\pi}{6}}\overset{EULER}{\ge}$$

$$\ge(28r^2+4\cdot 2r\cdot r)\cdot\frac{1}{\frac{1}{2}}=2(28r^2+8r^2)=2\cdot 36r^2=72r^2$$

Equality holds for $a=b=c$.

4190. In ΔABC the following relationship holds:

$$\frac{a}{h_a\sin A}+\frac{b}{h_b\sin B}+\frac{c}{h_c\sin C}\ge 4$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a\sin A}+\frac{b}{h_b\sin B}+\frac{c}{h_c\sin C}=\sum_{cyc}\frac{a}{h_a\sin A}=$$

$$=\sum_{cyc}\frac{a}{\frac{2F}{a}\cdot\frac{a}{2R}}=\frac{R}{F}\sum_{cyc}a=\frac{R}{F}\cdot 2s=\frac{R}{rs}\cdot 2s=\frac{2R}{r}\overset{EULER}{\ge}\frac{2\cdot 2r}{r}=4$$

Equality holds for $a=b=c$.

4191. In ΔABC the following relationship holds:

$$\frac{a}{h_a\sin\frac{A}{2}}+\frac{b}{h_b\sin\frac{B}{2}}+\frac{c}{h_c\sin\frac{C}{2}}\ge 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a\sin\frac{A}{2}}+\frac{b}{h_b\sin\frac{B}{2}}+\frac{c}{h_c\sin\frac{C}{2}}=\sum_{cyc}\frac{a}{h_a\sin\frac{A}{2}}=\sum_{cyc}\frac{a}{\frac{2F}{a}\sin\frac{A}{2}}=\frac{1}{2F}\sum_{cyc}\frac{a^2}{\sin\frac{A}{2}}\overset{CEBYSEV}{\ge}$$

$$\geq \frac{1}{2F}\cdot\frac{1}{3}\sum_{cyc} a^2 \sum_{cyc}\frac{1}{\sin\frac{A}{2}} \overset{JENSEN}{\geq} \frac{1}{6F}\cdot\sum_{cyc} a^2 \cdot \frac{3}{\sin\left(\frac{A+B+C}{6}\right)} = \frac{1}{2F}\sum_{cyc} a^2 \cdot \frac{1}{\frac{1}{2}} =$$

$$= \frac{1}{rs}\cdot 2(s^2 - r^2 - 4Rr) \overset{GERRETSEN}{\geq} \frac{2}{rs}(16Rr - 5r^2 - r^2 - 4Rr) = \frac{2(12Rr - 6r^2)}{rs}$$

$$= \frac{2(12R-6r)}{s} = \frac{12(2R-r)}{s} \geq 4\sqrt{3} \Leftrightarrow\Leftrightarrow 3(2R-r) \geq s\sqrt{3} \quad \text{(to prove)}$$

$$s\sqrt{3} \overset{DOUCET}{\leq} 4R + r \leq 3(2R-r),\ 4R + r \leq 6R - 3r,\ R \geq 2r \quad \text{(Euler)}$$

Equality holds for $a = b = c$.

4192. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin A}{h_b} \geq \frac{\sqrt{3}}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\frac{\sin A}{h_b} = \sum_{cyc}\frac{\frac{a}{2R}}{\frac{2F}{b}} = \frac{1}{4RF}\sum_{cyc} ab =$$

$$= \frac{1}{4Rrs}\cdot(s^2 + r^2 + 4Rr) \overset{MITRINOVIC}{\geq} \frac{1}{4Rr\cdot\frac{3\sqrt{3}}{2}R}(s^2 + r^2 + 4Rr) \geq$$

$$\overset{GERRETSEN}{\geq} \frac{1}{6\sqrt{3}R^2 r}(16Rr - 5r^2 + r^2 + 4Rr) = \frac{1}{6\sqrt{3}R^2 r}(20Rr - 4r^2) =$$

$$= \frac{2r(10R-2r)}{6\sqrt{3}R^2 r} = \frac{10R-2r}{3\sqrt{3}R^2} \geq \frac{\sqrt{3}}{R} \Leftrightarrow 10R - 2r \geq 9R \Leftrightarrow R \geq 2r \quad \text{(Euler)}$$

Equality holds for $a = b = c$.

4193. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin A}{h_a} \geq \frac{\sqrt{3}}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\frac{\sin A}{h_a}=\sum_{cyc}\frac{\frac{a}{2R}}{\frac{2F}{a}}=\frac{1}{4RF}\sum_{cyc}a^2=\frac{1}{4Rrs}\cdot 2(s^2-r^2-4Rr)\overset{MITRINOVIC}{\geq}$$

$$\geq\frac{1}{2Rr\cdot\frac{3\sqrt{3}R}{2}}(s^2-r^2-4Rr)=\frac{1}{3\sqrt{3}R^2r}(s^2-r^2-4Rr)\geq$$

$$\overset{GERRETSEN}{\geq}\frac{1}{3\sqrt{3}R^2r}(16Rr-5r^2-r^2-4Rr)=\frac{6r(2R-r)}{3\sqrt{3}R^2r}=$$

$$=\frac{2(2R-r)}{\sqrt{3}R^2}\geq\frac{\sqrt{3}}{R}\Leftrightarrow 2(2R-r)\geq 3r,\ 4R-2r\geq 3R\Leftrightarrow R\geq 2r\quad \text{(Euler)}$$

Equality holds for $a=b=c$.

4194. **In ΔABC the following relationship holds:**

$$\frac{ab}{\cos\frac{C}{2}}+\frac{bc}{\cos\frac{A}{2}}+\frac{ca}{\cos\frac{B}{2}}\geq 24\sqrt{3}r^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{ab}{\cos\frac{C}{2}}+\frac{bc}{\cos\frac{A}{2}}+\frac{ca}{\cos\frac{B}{2}}=\sum_{cyc}\frac{ab}{\cos\frac{C}{2}}\overset{CEBYSHEV}{\geq}\frac{1}{3}\sum_{cyc}ab\cdot\sum_{cyc}\frac{1}{\cos\frac{C}{2}}\overset{JENSEN}{\geq}$$

$$\geq\frac{1}{3}\cdot\sum_{cyc}ab\cdot 3\cdot\frac{1}{\cos\left(\frac{A+B+C}{6}\right)}\overset{AM-GM}{\geq}\frac{1}{\cos\frac{\pi}{6}}\cdot\sum_{cyc}ab\geq\frac{1}{\frac{\sqrt{3}}{2}}\cdot 3\sqrt[3]{(abc)^2}=$$

$$=\frac{6}{\sqrt{3}}\cdot\sqrt[3]{(4RF)^2}=\frac{6\sqrt{3}}{3}\cdot\sqrt[3]{(4Rrs)^2}\overset{EULER}{\geq}2\sqrt{3}\cdot\sqrt[3]{(8r^2s)^2}\geq$$

$$\overset{MITRINOVIC}{\geq}2\sqrt{3}\cdot\sqrt[3]{\left(8r^2\cdot 3\sqrt{3}r\right)^2}=2\sqrt{3}\cdot\sqrt[3]{64r^6\cdot 27}=2\sqrt{3}\cdot 4\cdot 3\cdot r^2=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.

4195. **In ΔABC the following relationship holds:**

$$\frac{a\sin A}{h_a}+\frac{b\sin B}{h_b}+\frac{c\sin C}{h_c}\geq 3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a\sin A}{h_a}+\frac{b\sin B}{h_b}+\frac{c\sin C}{h_c}=\sum_{cyc}\frac{a\sin A}{h_a}=$$

$$=\sum_{cyc}\frac{a\cdot\frac{a}{2R}}{\frac{2F}{a}}=\frac{1}{4RF}\sum_{cyc}a^3\overset{AM-GM}{\geq}\frac{1}{abc}\cdot 3\sqrt[3]{(abc)^3}=\frac{3abc}{abc}=3$$

Equality holds for $a=b=c$.

4196. In ΔABC the following relationship holds:

$$\frac{ab}{\sin C}+\frac{bc}{\sin A}+\frac{ca}{\sin B}\geq 24\sqrt{3}r^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{ab}{\sin C}+\frac{bc}{\sin A}+\frac{ca}{\sin B}=\sum_{cyc}\frac{ab}{\sin C}=\sum_{cyc}\frac{ab}{\frac{c}{2R}}=2R\sum\frac{ab}{c}\overset{AM-GM}{\geq}2R\cdot 3\sqrt[3]{\frac{ab}{c}\cdot\frac{bc}{a}\cdot\frac{ca}{b}}=$$

$$=6R\sqrt[3]{abc}=6R\sqrt[3]{4Rrs}\overset{EULER}{\geq}6R\cdot\sqrt[3]{4\cdot 2r\cdot rs}=12R\sqrt[3]{r^2s}\overset{MITRINOVIC}{\geq}$$

$$\geq 12R\sqrt[3]{r^2\cdot 3\sqrt{3}r}\overset{EULER}{\geq}12\cdot 2r\cdot\sqrt[3]{\left(r\sqrt{3}\right)^3}=24r\cdot\sqrt{3}r=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.

4197. In ΔABC the following relationship holds:

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq 6\sqrt{3}r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{ab}{c}\cdot\frac{bc}{a}\cdot\frac{ca}{b}}=3\sqrt[3]{abc}=3\sqrt[3]{4Rrs}\overset{EULER}{\geq}$$

$$\geq 3\sqrt[3]{4\cdot 2r\cdot rs}\overset{MITRINOVIC}{\geq}3\cdot 2\sqrt[3]{r^2\cdot 3\sqrt{3}r}=6\sqrt[3]{\left(r\sqrt{3}\right)^3}=6\sqrt{3}r$$

Equality holds for $a=b=c$.

4198. **In acute ΔABC the following relationship holds:**

$$\frac{a}{h_a\cos^2\frac{A}{2}}+\frac{b}{h_b\cos^2\frac{B}{2}}+\frac{c}{h_c\cos^2\frac{C}{2}}\ge\frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a\cos^2\frac{A}{2}}+\frac{b}{h_b\cos^2\frac{B}{2}}+\frac{c}{h_c\cos^2\frac{C}{2}}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{abc}{h_ah_bh_c\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)^2}}=$$

$$=3\sqrt[3]{\frac{abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\cdot\left(\frac{s}{4R}\right)^2}}=3\sqrt[3]{\frac{(abc)^2\cdot16R^2}{8F^3\cdot s^2}}=3\sqrt[3]{\frac{16R^2F^2\cdot2R^2}{F^3s^2}}=$$

$$=3\sqrt[3]{\frac{32R^4}{Fs^2}}\overset{EULER}{\ge}3\sqrt[3]{\frac{32\cdot2r\cdot R^3}{rs^3}}=3\sqrt[3]{64\left(\frac{R}{s}\right)^3}=\frac{12R}{s}\overset{MITRINOVIC}{\ge}$$

$$\ge\frac{12\cdot\frac{2}{3\sqrt{3}}s}{s}=\frac{24}{3\sqrt{3}}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}$$

Equality holds for $a=b=c$.

4199. **In acute ΔABC the following relationship holds:**

$$\frac{a}{h_a\cos^2\frac{A}{2}}+\frac{b}{h_b\cos^2\frac{B}{2}}+\frac{c}{h_c\cos^2\frac{C}{2}}\ge\frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a\cos^2\frac{A}{2}}+\frac{b}{h_b\cos^2\frac{B}{2}}+\frac{c}{h_c\cos^2\frac{C}{2}}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{abc}{h_ah_bh_c\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)^2}}=$$

$$= 3\sqrt[3]{\frac{abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\cdot\left(\frac{s}{4R}\right)^2}} = 3\sqrt[3]{\frac{(abc)^2\cdot 16R^2}{8F^3\cdot s^2}} = 3\sqrt[3]{\frac{16R^2F^2\cdot 2R^2}{F^3s^2}} =$$

$$= 3\sqrt[3]{\frac{32R^4}{Fs^2}} \overset{EULER}{\geq} 3\sqrt[3]{\frac{32\cdot 2r\cdot R^3}{rs^3}} = 3\sqrt[3]{64\left(\frac{R}{s}\right)^3} = \frac{12R}{s} \overset{MITRINOVIC}{\geq}$$

$$\geq \frac{12\cdot\frac{2}{3\sqrt{3}}s}{s} = \frac{24}{3\sqrt{3}} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$$

Equality holds for $a = b = c$.

4200. In acute ΔABC the following relationship holds:

$$\frac{a}{r_a\cos A} + \frac{b}{r_b\cos B} + \frac{c}{r_c\cos C} \geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

We will use the next known result:

$$\cos A\cos B\cos C \leq \frac{1}{8} \quad (1)$$

$$\frac{a}{r_a\cos A} + \frac{b}{r_b\cos B} + \frac{c}{r_c\cos C} \geq 3\sqrt[3]{\frac{abc}{r_ar_br_c\cos A\cos B\cos C}} \geq$$

$$\overset{(1)}{\geq} 3\sqrt[3]{\frac{abc}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\cdot\frac{1}{8}}} = 3\sqrt[3]{\frac{8abc(s-a)(s-b)(s-c)}{F^3}} =$$

$$= 3\sqrt[3]{\frac{8abc\cdot F^2}{sF^3}} = 6\sqrt[3]{\frac{4RF}{sF}} = 6\sqrt[3]{\frac{4R}{s}} \overset{MITRINOVIC}{\geq} 6\sqrt[3]{\frac{4\cdot\frac{2}{3\sqrt{3}}s}{s}} = 6\sqrt[3]{\left(\frac{2}{\sqrt{3}}\right)^3} =$$

$$= 6\cdot\frac{2}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$$

Equality holds for $a = b = c$.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru