On a Centroid Problem: Weighted, Spatial, and Higher-Dimensional Extensions

Mihály Bencze
Braşov, Romania
benczemihaly@gmail.com

Abstract

We begin with an elementary quadrilateral problem proposed by Jafar Nikpour. Four centroids associated with a fixed point and the consecutive sides of a quadrilateral form a parallelogram whose sides are parallel to the diagonals of the original quadrilateral, and the two areas occur in the ratio $2:9$. We give a self-contained proof of the original result and place it in a wider affine framework. The ordinary centroids are replaced by weighted barycenters, the four vertices are allowed to be noncoplanar, and the planar area identity is replaced by a vector-area identity. We also establish a higher-dimensional Gram-determinant formula, characterize the unequal-weight case, extend the construction to cyclic n-gons, and derive applications to tetrahedral sections, projected areas, and pyramid volumes.

2020 Mathematics Subject Classification. Primary 51M04; Secondary 51M25, 51N20.

Keywords. centroid; weighted barycenter; quadrilateral; parallelogram; vector area; skew quadrilateral; Gram determinant; tetrahedron.

1 Introduction

Centroid constructions are especially well suited to vector and affine methods: a barycenter is an affine combination, and differences of affine combinations often reveal hidden parallelisms. Classical examples include the Varignon parallelogram and related midpoint and centroid configurations; see, for example, [3, 4, 6]. General background on affine and Euclidean vector methods may be found in [2, 5].

One should distinguish the centroid of a finite system of vertices from the laminar centroid of the polygonal region. These notions agree for every triangle but not for a general quadrilateral; a useful discussion of this distinction appears in [1]. The present paper concerns centroids and weighted barycenters of finite point systems.

Our starting point is the problem reproduced in Figure 1, whose source diagram attributes it to Jafar Nikpour [7]. No publication data accompanied the supplied diagram. The purpose of this note is to give the original problem a complete vector proof and then derive systematic planar, spatial, and higher-dimensional extensions.

The principal observations are the following. Equal weights on two consecutive vertices cause the common terms to cancel, leaving the two diagonal vectors of the original quadrilateral. This remains true in every affine dimension. In three-dimensional space, even a skew quadrilateral therefore produces a planar parallelogram. Its area is controlled by a cross product, while in $\mathbb{R}^d$ the corresponding formula is expressed by a Gram determinant. We also show that equality of the two consecutive weights is necessary for the construction to work for every quadrilateral.

2 The original problem of Jafar Nikpour

Problem 1 (Nikpour)**.** Let P, A_1, A_2, A_3, A_4 be points in a plane, with $A_1A_2A_3A_4$ a simple quadrilateral. Let G_1, G_2, G_3, G_4 be the centroids of

$$\triangle PA_1A_2, \qquad \triangle PA_2A_3, \qquad \triangle PA_3A_4, \qquad \triangle PA_4A_1,$$

respectively. Prove that

$$G_1G_2 \parallel G_3G_4 \parallel A_1A_3,$$
$$G_2G_3 \parallel G_4G_1 \parallel A_2A_4,$$

and

$$2\,\mathrm{Area}(A_1A_2A_3A_4) = 9\,\mathrm{Area}(G_1G_2G_3G_4).$$

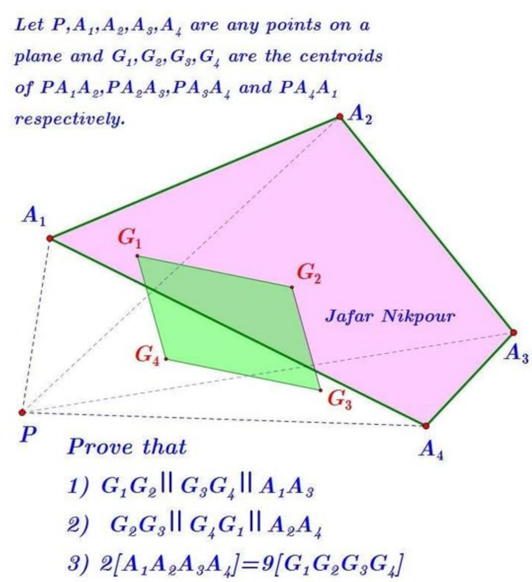

Figure 1: The supplied formulation of the original problem, attributed in the diagram to Jafar Nikpour.

Proof. Let $\mathbf{p}, \mathbf{a}_i, \mathbf{g}_i$ denote the position vectors of P, A_i, G_i, respectively, and read indices cyclically. Since G_i is the centroid of $\triangle PA_iA_{i+1}$,

$$\mathbf{g}_i = \frac{\mathbf{p} + \mathbf{a}_i + \mathbf{a}_{i+1}}{3}.$$

Consequently,

$$\overrightarrow{G_1G_2} = \mathbf{g}_2 - \mathbf{g}_1 = \frac{1}{3}(\mathbf{a}_3 - \mathbf{a}_1) = \frac{1}{3}\overrightarrow{A_1A_3},$$
$$\overrightarrow{G_2G_3} = \mathbf{g}_3 - \mathbf{g}_2 = \frac{1}{3}(\mathbf{a}_4 - \mathbf{a}_2) = \frac{1}{3}\overrightarrow{A_2A_4},$$
$$\overrightarrow{G_3G_4} = -\frac{1}{3}\overrightarrow{A_1A_3}, \qquad \overrightarrow{G_4G_1} = -\frac{1}{3}\overrightarrow{A_2A_4}.$$

These vector identities prove both parallelism assertions and show that $G_1G_2G_3G_4$ is a parallelogram.

Put

$$\mathbf{u} = \mathbf{a}_3 - \mathbf{a}_1, \qquad \mathbf{v} = \mathbf{a}_4 - \mathbf{a}_2.$$

For planar vectors, let $\det(\mathbf{u}, \mathbf{v})$ denote their oriented determinant. The shoelace formula, or a direct expansion, gives

$$2\,\mathrm{Area}(A_1A_2A_3A_4) = |\det(\mathbf{u}, \mathbf{v})|\,.$$

The adjacent sides of $G_1G_2G_3G_4$ are $\mathbf{u}/3$ and $\mathbf{v}/3$, and hence

$$\mathrm{Area}(G_1G_2G_3G_4) = \frac{1}{9}\,|\det(\mathbf{u}, \mathbf{v})|\,.$$

Eliminating the determinant yields

$$9\,\mathrm{Area}(G_1G_2G_3G_4) = 2\,\mathrm{Area}(A_1A_2A_3A_4),$$

which completes the proof. □

Remark 2.1. The vector equalities remain valid without any convexity assumption and even in degenerate cases. For a self-intersecting ordered quadrilateral, the area statement should be interpreted using oriented area.

3 The spatial generalized problem

Let P, A_1, A_2, A_3, A_4 be arbitrary points of $\mathbb{R}^3$. The four points A_1, A_2, A_3, A_4 are *not* assumed to be coplanar. Choose real numbers α, β such that

$$s = \alpha + 2\beta \neq 0, \qquad c = \frac{\beta}{s}.$$

Read subscripts cyclically, so $A_5 = A_1$, and define

$$\mathbf{g}_i = \frac{\alpha\mathbf{p} + \beta\mathbf{a}_i + \beta\mathbf{a}_{i+1}}{s}, \qquad i = 1, 2, 3, 4. \tag{1}$$

Thus G_i is the weighted barycenter of P, A_i, A_{i+1}, with respective weights α, β, β. When $\alpha = \beta = 1$, it is the ordinary centroid of $\triangle PA_iA_{i+1}$.

Define the two diagonal direction vectors

$$\mathbf{u} = \mathbf{a}_3 - \mathbf{a}_1, \qquad \mathbf{v} = \mathbf{a}_4 - \mathbf{a}_2. \tag{2}$$

Spatial theorem

The points G_1, G_2, G_3, G_4 satisfy

$$\begin{aligned} \overrightarrow{G_1G_2} &= c\mathbf{u}, & \overrightarrow{G_2G_3} &= c\mathbf{v}, \\ \overrightarrow{G_3G_4} &= -c\mathbf{u}, & \overrightarrow{G_4G_1} &= -c\mathbf{v}. \end{aligned} \tag{3}$$

Consequently:

1. $G_1G_2G_3G_4$ is a planar parallelogram, possibly degenerate, even when $A_1A_2A_3A_4$ is skew.
2. Whenever the relevant segments are nonzero, its sides are parallel to the two spatial diagonals:

$$G_1G_2 \parallel G_3G_4 \parallel A_1A_3, \qquad G_2G_3 \parallel G_4G_1 \parallel A_2A_4.$$

3. Its scalar area is

$$\boxed{\mathrm{Area}(G_1G_2G_3G_4) = c^2 \left\|\mathbf{u} \times \mathbf{v}\right\| = \frac{\beta^2}{(\alpha+2\beta)^2} \left\|(\mathbf{a}_3 - \mathbf{a}_1) \times (\mathbf{a}_4 - \mathbf{a}_2)\right\|.} \tag{4}$$

Proof of the side relations and coplanarity

Subtracting two consecutive formulas in (1) makes the common term $\alpha\mathbf{p}$ cancel. For example,

$$\begin{aligned} \mathbf{g}_2 - \mathbf{g}_1 &= \frac{\alpha\mathbf{p} + \beta\mathbf{a}_2 + \beta\mathbf{a}_3 - (\alpha\mathbf{p} + \beta\mathbf{a}_1 + \beta\mathbf{a}_2)}{s} \\ &= \frac{\beta}{s}(\mathbf{a}_3 - \mathbf{a}_1) = c\mathbf{u}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{g}_3 - \mathbf{g}_2 &= c(\mathbf{a}_4 - \mathbf{a}_2) = c\mathbf{v}, \\ \mathbf{g}_4 - \mathbf{g}_3 &= c(\mathbf{a}_1 - \mathbf{a}_3) = -c\mathbf{u}, \\ \mathbf{g}_1 - \mathbf{g}_4 &= c(\mathbf{a}_2 - \mathbf{a}_4) = -c\mathbf{v}. \end{aligned}$$

This proves (3). In particular,

$$\mathbf{g}_1 + \mathbf{g}_3 = \mathbf{g}_2 + \mathbf{g}_4, \tag{5}$$

so the two diagonals of $G_1G_2G_3G_4$ have the same midpoint. Equation (5) is valid in $\mathbb{R}^3$, and therefore the four G_i's form a parallelogram and are automatically coplanar.

If $c \neq 0$ and $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, the plane of this parallelogram has normal vector

$$\mathbf{n} = \mathbf{u} \times \mathbf{v}$$

and has the explicit equation

$$\boxed{\mathbf{n} \cdot (\mathbf{x} - \mathbf{g}_1) = 0.} \tag{6}$$

Finally, the area of a parallelogram generated by the adjacent vectors $c\mathbf{u}$ and $c\mathbf{v}$ is

$$\left\|(c\mathbf{u}) \times (c\mathbf{v})\right\| = c^2 \left\|\mathbf{u} \times \mathbf{v}\right\|,$$

which proves (4).

4 The correct notion of area for a skew quadrilateral

A skew quadrilateral does not bound a planar region, so it has no ordinary planar area. The appropriate invariant is its *vector area*. For an ordered spatial polygon $X_1X_2 \cdots X_n$, define

$$\boldsymbol{\mathcal{A}}(X_1 \cdots X_n) = \frac{1}{2}\sum_{i=1}^{n} \mathbf{x}_i \times \mathbf{x}_{i+1}, \qquad X_{n+1} = X_1. \tag{7}$$

This vector is independent of the choice of origin. Its component in a unit direction $\mathbf{e}$ is the oriented area of the orthogonal projection of the polygon onto the plane perpendicular to $\mathbf{e}$.

For the ordered quadrilateral $A_1A_2A_3A_4$, expansion gives

$$\begin{aligned} 2\boldsymbol{\mathcal{A}}(A_1A_2A_3A_4) &= \mathbf{a}_1 \times \mathbf{a}_2 + \mathbf{a}_2 \times \mathbf{a}_3 + \mathbf{a}_3 \times \mathbf{a}_4 + \mathbf{a}_4 \times \mathbf{a}_1 \\ &= (\mathbf{a}_3 - \mathbf{a}_1) \times (\mathbf{a}_4 - \mathbf{a}_2) \\ &= \mathbf{u} \times \mathbf{v}. \end{aligned} \tag{8}$$

Because $G_1G_2G_3G_4$ is a parallelogram, its vector area is the cross product of two adjacent side vectors:

$$\begin{aligned} \boldsymbol{\mathcal{A}}(G_1G_2G_3G_4) &= (\mathbf{g}_2 - \mathbf{g}_1) \times (\mathbf{g}_4 - \mathbf{g}_1) \\ &= (c\mathbf{u}) \times (c\mathbf{v}) = c^2\mathbf{u} \times \mathbf{v}. \end{aligned} \tag{9}$$

Combining (8) and (9) yields the spatial area identity

$$\boxed{(\alpha + 2\beta)^2\boldsymbol{\mathcal{A}}(G_1G_2G_3G_4) = 2\beta^2\boldsymbol{\mathcal{A}}(A_1A_2A_3A_4).} \tag{10}$$

If the A_i's are coplanar and listed around a simple quadrilateral, then $\|\boldsymbol{\mathcal{A}}(A_1A_2A_3A_4)\|$ is its ordinary area. Thus (10) reduces to

$$(\alpha + 2\beta)^2 \operatorname{Area}(G_1G_2G_3G_4) = 2\beta^2 \operatorname{Area}(A_1A_2A_3A_4). \tag{11}$$

For ordinary centroids, $\alpha = \beta = 1$, and (11) becomes exactly

$$9 \operatorname{Area}(G_1G_2G_3G_4) = 2 \operatorname{Area}(A_1A_2A_3A_4).$$

5 Sharp geometric refinements

Assume in this section that $c \neq 0$ and $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, so the new parallelogram is nondegenerate. Since its adjacent side vectors are $c\mathbf{u}$ and $c\mathbf{v}$, all of its metric properties can be read directly from the two diagonal directions of the original spatial quadrilateral.

Lengths, angle, and area

$$\begin{aligned} |G_1G_2| &= |c| \; \|\mathbf{u}\| \,, \\ |G_2G_3| &= |c| \; \|\mathbf{v}\| \,, \\ \cos \angle G_2G_1G_4 &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \\ \operatorname{Area}(G_1G_2G_3G_4) &= c^2 \, \|\mathbf{u}\| \, \|\mathbf{v}\| \sin \theta, \end{aligned} \tag{12}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Complete shape criteria

$$\begin{array}{c|c} \text{Shape of } G_1G_2G_3G_4 & \text{Necessary and sufficient condition} \\ \hline \text{rectangle} & \mathbf{u}\cdot\mathbf{v}=0 \\ \text{rhombus} & \|\mathbf{u}\|=\|\mathbf{v}\| \\ \text{square} & \mathbf{u}\cdot\mathbf{v}=0 \text{ and } \|\mathbf{u}\|=\|\mathbf{v}\| \end{array} \tag{13}$$

The construction degenerates to a segment or a point precisely when

$$\beta = 0 \quad \text{or} \quad \mathbf{u}\times\mathbf{v}=\mathbf{0}. \tag{14}$$

Indeed, $\beta = 0$ gives $G_1 = \cdots = G_4 = P$, while $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ means that the two side directions are linearly dependent.

6 Why the two adjacent weights must be equal

The equal weights on A_i and A_{i+1} are not an accidental choice. Consider the more general cyclic affine construction

$$\mathbf{h}_i = \frac{\alpha\mathbf{p}+\beta\mathbf{a}_i+\gamma\mathbf{a}_{i+1}}{\alpha+\beta+\gamma}, \qquad \alpha+\beta+\gamma\neq 0. \tag{15}$$

A quadrilateral is a parallelogram exactly when its diagonals have the same midpoint, that is, $\mathbf{h}_1+\mathbf{h}_3 = \mathbf{h}_2+\mathbf{h}_4$. Direct calculation gives

$$\mathbf{h}_1+\mathbf{h}_3-\mathbf{h}_2-\mathbf{h}_4 = \frac{\beta-\gamma}{\alpha+\beta+\gamma}(\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_4). \tag{16}$$

Therefore:

- For a fixed quadrilateral, the H_i's form a parallelogram exactly when
$$(\beta-\gamma)(\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_4)=\mathbf{0}.$$
- The construction works for *every* choice of the four vertices if and only if $\beta=\gamma$.
- If $\beta\neq\gamma$, it still works when $\mathbf{a}_1+\mathbf{a}_3=\mathbf{a}_2+\mathbf{a}_4$, namely when the original A-quadrilateral is itself a parallelogram.

This proves that equal adjacent weights are the unique universal choice producing the spatial parallelogram theorem.

7 Extension to every Euclidean dimension

The side-vector proof used only affine subtraction, so it remains valid in $\mathbb{R}^d$ for every $d\geq 2$. Cross products do not exist in general dimensions, but the area of the parallelogram generated by $\mathbf{u},\mathbf{v}\in\mathbb{R}^d$ is determined by the Gram determinant [8]:

$$\mathrm{Area}_2(\mathbf{u},\mathbf{v}) = \sqrt{\det\begin{pmatrix}\mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v}\\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v}\end{pmatrix}} = \sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}. \tag{17}$$

Hence the complete $\mathbb{R}^d$ area formula is

$$\boxed{\mathrm{Area}_2(G_1G_2G_3G_4) = c^2\sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2}.} \tag{18}$$

Formula (18) follows because both generating vectors are multiplied by c, so the 2-dimensional area is multiplied by c^2. For $d = 3$, the square root in (18) is $\|\mathbf{u}\times\mathbf{v}\|$.

8 Cyclic n-gon generalization

Let $A_1, \ldots, A_n \in \mathbb{R}^d$, with cyclic indices, and let $\mathbf{q} \in \mathbb{R}^d$ and $c \in \mathbb{R}$ be fixed. Define

$$\mathbf{g}_i = \mathbf{q} + c(\mathbf{a}_i + \mathbf{a}_{i+1}), \qquad i = 1, \ldots, n. \tag{19}$$

The weighted-barycenter construction is the special case

$$\mathbf{q} = \frac{\alpha}{\alpha + 2\beta}\mathbf{p}, \qquad c = \frac{\beta}{\alpha + 2\beta}.$$

For every i,

$$\boxed{\overrightarrow{G_iG_{i+1}} = c(\mathbf{a}_{i+2} - \mathbf{a}_i) = c\,\overrightarrow{A_iA_{i+2}}.} \tag{20}$$

Thus the sides of the derived n-gon are scaled copies of the second-neighbor chords of the original n-gon. In particular,

$$\mathrm{Perimeter}(G_1\cdots G_n) = |c|\sum_{i=1}^{n}|A_iA_{i+2}|. \tag{21}$$

In $\mathbb{R}^3$, its vector area can also be calculated exactly. Using (7), translation invariance, and (19),

$$\begin{aligned}\boldsymbol{\mathcal{A}}(G_1\cdots G_n) &= \frac{c^2}{2}\sum_{i=1}^{n}(\mathbf{a}_i + \mathbf{a}_{i+1}) \times (\mathbf{a}_{i+1} + \mathbf{a}_{i+2}) \\ &= 2c^2\boldsymbol{\mathcal{A}}(A_1\cdots A_n) + \frac{c^2}{2}\sum_{i=1}^{n}\mathbf{a}_i \times \mathbf{a}_{i+2}.\end{aligned} \tag{22}$$

For $n = 4$, the last sum vanishes in opposite pairs:

$$\mathbf{a}_1 \times \mathbf{a}_3 + \mathbf{a}_3 \times \mathbf{a}_1 + \mathbf{a}_2 \times \mathbf{a}_4 + \mathbf{a}_4 \times \mathbf{a}_2 = \mathbf{0}.$$

Therefore (22) reduces to the quadrilateral identity (10).

9 Applications

9.1 A midpoint section of a tetrahedron

Let A_1, A_2, A_3, A_4 be the vertices of a nondegenerate tetrahedron, and let M_i be the midpoint of the cyclic edge A_iA_{i+1}. This is (1) with $\alpha = 0$, $\beta = 1$, and $c = 1/2$. Hence the four edge midpoints are coplanar and form a parallelogram:

$$\begin{aligned}\overrightarrow{M_1M_2} &= \frac{1}{2}\overrightarrow{A_1A_3},\\ \overrightarrow{M_2M_3} &= \frac{1}{2}\overrightarrow{A_2A_4}.\end{aligned} \tag{23}$$

Its area and plane are

$$\operatorname{Area}(M_1M_2M_3M_4) = \frac{1}{4}\left\|\overrightarrow{A_1A_3}\times\overrightarrow{A_2A_4}\right\|, \tag{24}$$

$$\left(\overrightarrow{A_1A_3}\times\overrightarrow{A_2A_4}\right)\cdot\left(\mathbf{x}-\frac{\mathbf{a}_1+\mathbf{a}_2}{2}\right) = 0. \tag{25}$$

This parallelogram is the intersection of the tetrahedron with the plane through those four midpoints.

9.2 The ordinary-centroid construction in space

Taking $\alpha = \beta = 1$ gives $c = 1/3$. Therefore, even for a skew quadrilateral,

$$\begin{gathered}\overrightarrow{G_1G_2} = \frac{1}{3}\overrightarrow{A_1A_3}, \qquad \overrightarrow{G_2G_3} = \frac{1}{3}\overrightarrow{A_2A_4},\\ \operatorname{Area}(G_1G_2G_3G_4) = \frac{1}{9}\left\|\overrightarrow{A_1A_3}\times\overrightarrow{A_2A_4}\right\|, \quad 9\boldsymbol{\mathcal{A}}(G) = 2\boldsymbol{\mathcal{A}}(A).\end{gathered} \tag{26}$$

Taking the dot product of the last identity with any unit vector $\mathbf{e}$ shows that the oriented projected areas satisfy the same ratio $2 : 9$ in every viewing direction. This is useful when only orthogonal projections of a spatial polygon are available.

9.3 Volume of the pyramid with apex P

Assume that $G_1G_2G_3G_4$ is nondegenerate and use it as the base of a pyramid with apex P. Since

$$\mathbf{g}_1 - \mathbf{p} = \frac{\beta}{\alpha+2\beta}(\mathbf{a}_1+\mathbf{a}_2-2\mathbf{p}) = c(\mathbf{a}_1+\mathbf{a}_2-2\mathbf{p}),$$

and its oriented base-area vector is

$$(c\mathbf{u})\times(c\mathbf{v}) = c^2(\mathbf{u}\times\mathbf{v}),$$

the scalar triple-product formula gives

$$\boxed{\operatorname{Vol}(P-G_1G_2G_3G_4) = \frac{|c|^3}{3}\left|(\mathbf{a}_1+\mathbf{a}_2-2\mathbf{p})\cdot(\mathbf{u}\times\mathbf{v})\right|.} \tag{27}$$

As a useful special case, let $P = A_1$, and let $V_T = \operatorname{Vol}(A_1A_2A_3A_4)$. Put

$$\mathbf{b} = \mathbf{a}_2-\mathbf{a}_1, \quad \mathbf{d} = \mathbf{a}_3-\mathbf{a}_1, \quad \mathbf{e} = \mathbf{a}_4-\mathbf{a}_1.$$

Then $\mathbf{u} = \mathbf{d}$, $\mathbf{v} = \mathbf{e}-\mathbf{b}$, and

$$\begin{aligned}|(\mathbf{a}_2-\mathbf{a}_1)\cdot[(\mathbf{a}_3-\mathbf{a}_1)\times(\mathbf{a}_4-\mathbf{a}_2)]| &= |\mathbf{b}\cdot[\mathbf{d}\times(\mathbf{e}-\mathbf{b})]|\\ &= |\mathbf{b}\cdot(\mathbf{d}\times\mathbf{e})|\\ &= 6V_T.\end{aligned}$$

Therefore,

$$\boxed{\mathrm{Vol}(A_1 - G_1G_2G_3G_4) = 2|c|^3 V_T.} \tag{28}$$

For the midpoint section $c = 1/2$, this volume is $V_T/4$. For the ordinary-centroid construction $c = 1/3$, it is $2V_T/27$.

10 A fully calculated spatial example

Take

$$\begin{aligned} P &= (1,-1,2), \quad A_1 = (0,0,0), \qquad A_2 = (2,0,1), \\ A_3 &= (1,3,0), \qquad A_4 = (-1,1,4), \qquad \alpha = 2, \quad \beta = 1. \end{aligned}$$

Here $s = 4$ and $c = 1/4$. Formula (1) gives

$$\begin{aligned} G_1 &= \frac{2P + A_1 + A_2}{4} = \left(1, -\frac{1}{2}, \frac{5}{4}\right), \\ G_2 &= \frac{2P + A_2 + A_3}{4} = \left(\frac{5}{4}, \frac{1}{4}, \frac{5}{4}\right), \\ G_3 &= \frac{2P + A_3 + A_4}{4} = \left(\frac{1}{2}, \frac{1}{2}, 2\right), \\ G_4 &= \frac{2P + A_4 + A_1}{4} = \left(\frac{1}{4}, -\frac{1}{4}, 2\right). \end{aligned} \tag{29}$$

The two original diagonal vectors are

$$\mathbf{u} = A_3 - A_1 = (1,3,0), \qquad \mathbf{v} = A_4 - A_2 = (-3,1,3). \tag{30}$$

Direct subtraction confirms

$$\begin{aligned} G_2 - G_1 &= \left(\frac{1}{4}, \frac{3}{4}, 0\right) = \frac{1}{4}\mathbf{u}, \\ G_3 - G_2 &= \left(-\frac{3}{4}, \frac{1}{4}, \frac{3}{4}\right) = \frac{1}{4}\mathbf{v}, \\ G_4 - G_3 &= -\frac{1}{4}\mathbf{u}, \qquad G_1 - G_4 = -\frac{1}{4}\mathbf{v}. \end{aligned}$$

Thus the four computed points do form a parallelogram.

Moreover,

$$\mathbf{u} \cdot \mathbf{v} = (1)(-3) + (3)(1) + (0)(3) = 0,$$

so the parallelogram is a rectangle. It is not a square because

$$\|\mathbf{u}\| = \sqrt{10}, \qquad \|\mathbf{v}\| = \sqrt{19}.$$

Its normal vector and area are

$$\mathbf{u} \times \mathbf{v} = (9,-3,10), \qquad \mathrm{Area}(G) = \frac{1}{16}\sqrt{9^2 + (-3)^2 + 10^2} = \boxed{\frac{\sqrt{190}}{16}}. \tag{31}$$

Using $G_1 = (1, -1/2, 5/4)$, equation (6) becomes

$$(9, -3, 10) \cdot \left((x, y, z) - \left(1, -\frac{1}{2}, \frac{5}{4}\right) \right) = 0,$$

or, after simplification,

$$\boxed{9x - 3y + 10z = 23}. \tag{32}$$

Substitution of each point in (29) verifies that all four lie in this plane.

11 Summary of the main formulas

With

$$c = \frac{\beta}{\alpha + 2\beta}, \qquad \mathbf{u} = A_3 - A_1, \qquad \mathbf{v} = A_4 - A_2,$$

the entire construction is governed by

$$\boxed{\begin{array}{cc} \overrightarrow{G_1G_2} = c\mathbf{u}, & \overrightarrow{G_2G_3} = c\mathbf{v}, \\ \mathrm{Area}(G) = c^2 \|\mathbf{u} \times \mathbf{v}\|, & \boldsymbol{\mathcal{A}}(G) = 2c^2 \boldsymbol{\mathcal{A}}(A), \\ \text{plane normal} = \mathbf{u} \times \mathbf{v}, & \text{square condition} : \mathbf{u} \cdot \mathbf{v} = 0,\ \|\mathbf{u}\| = \|\mathbf{v}\|. \end{array}}$$

These formulas show that the original planar theorem is an affine and metric shadow of a more general spatial phenomenon: equal weights on consecutive vertices convert the two diagonal directions of any ordered spatial quadrilateral into the two side directions of a planar parallelogram.

Acknowledgement

The author acknowledges Jafar Nikpour as the proposer of the original problem that motivated this article.

References

[1] S. András and C. Tamási, *Teaching centroids in theory and in practice*, Teaching Mathematics and Computer Science **11** (2013), no. 1, 67–88. https://doi.org/10.5485/TMCS.2013.0332

[2] M. Berger, *Geometry I*, Springer-Verlag, Berlin, 1987.

[3] H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley & Sons, New York, 1969.

[4] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, Washington, DC, 1967.

[5] J. Gallier, *Geometric Methods and Applications: For Computer Science and Engineering*, 2nd ed., Texts in Applied Mathematics, vol. 38, Springer, New York, 2011.

[6] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Mathematical Association of America, Washington, DC, 1995.

[7] J. Nikpour, *A centroid problem for a quadrilateral*, undated problem diagram circulated online; source image supplied to the present author.

[8] G. Strang, *Linear Algebra and Its Applications*, 4th ed., Brooks/Cole, Belmont, CA, 2006.