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NEW IDENTITIES AND INEQUALITIES IN TRIANGLE-(V)

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We consider a nondegenerate Euclidean triangle ABC. We use the usual notation: a, b, c are the side lengths; s is the semiperimeter; Δ is the area; r and R are the inradius and the circumradius; h_a, h_b, h_c are the altitudes; r_a, r_b, r_c are the exradii; and n_a, n_b, n_c are the Nagel cevians; l_a, l_b, l_c are angle bisectors of a triangle.

We put:

$$q_a = \sqrt{(b-c)^2 + 4r^2}, \qquad q_b = \sqrt{(c-a)^2 + 4r^2}, \qquad q_c = \sqrt{(a-b)^2 + 4r^2}.$$

For an integer m, we consider the inequality:

$$\frac{3(n_a^2+n_b^2+n_c^2)}{r^2} \le \left(\frac{n_a}{n_a-q_a} + \frac{n_b}{n_b-q_b} + \frac{n_c}{n_c-q_c}\right)^m \frac{R}{2r}. \tag{1}$$

The problem is to determine the least integer exponent m for which (1) is true for every nondegenerate triangle.

THEOREM.

The least possible integer exponent is:

$$\boxed{m_{opt} = 2}.$$

For m = 2, equality holds only for the equilateral triangle.

PROOF.

We use the following Nagel-cevian identity [1]:

$$\boxed{\frac{r_a}{r} = \frac{n_a}{n_a-q_a}} \tag{2}$$

and the identity:

$$\boxed{s^2 = n_a^2 + 2r_a h_a}, \tag{3}$$

together with their cyclic analogues.

From (2), since $\frac{r_a}{r} > 0$ and $n_a > 0$, we obtain $n_a - q_a > 0$. Therefore, all the denominators in (1) are strictly positive.

From (2), after cyclic summation, we obtain:

$$\sum_{cyc} \frac{n_a}{n_a-q_a} = \frac{r_a+r_b+r_c}{r}. \tag{4}$$

To calculate the sum of the exradii, we put:

$$x = s-a, \qquad y = s-b, \qquad z = s-c. \ x+y+z=s$$

Then:

$$r_a = \frac{\Delta}{x}, \qquad r_b = \frac{\Delta}{y}, \qquad r_c = \frac{\Delta}{z}, \qquad \Delta^2 = sxyz.$$

Also:

$$(x+y)(y+z)(z+x) = abc = 4Rrs,$$

but:

$$(x+y)(y+z)(z+x) = s(xy+yz+zx) - xyz$$

and:

$$\text{and } xyz = r^2 s. \text{ Therefore}$$

Therefore:

$$xy + yz + zx = 4Rr + r^2.$$

Consequently:

$$r_a + r_b + r_c = \Delta\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \Delta\frac{xy + yz + zx}{xyz} = 4R + r.$$

Using this result in (4), we obtain:

$$\boxed{\sum_{cyc} \frac{n_a}{n_a - q_a} = \frac{4R + r}{r}.} \tag{5}$$

Now, from (3), after cyclic summation, we have:

$$\mathbf{n_a^2 + n_b^2 + n_c^2 = 3s^2 - 2\sum_{cyc} r_a\, h_a.} \tag{6}$$

We calculate the last sum exactly. We shall use the elementary identities:

$$\mathbf{r_a + r_b + r_c = 4R + r,} \tag{7}$$

$$\mathbf{r_a r_b + r_b r_c + r_c r_a = s^2,} \tag{8}$$

$$\mathbf{r_a r_b r_c = rs^2.} \tag{9}$$

Indeed, for (8) we have:

$$r_a r_b + r_b r_c + r_c r_a = \Delta^2\left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right) = sxyz\frac{x + y + z}{xyz} = s^2.$$

and, for (9):

$$r_a r_b r_c = \frac{\Delta^3}{xyz} = \Delta s = rs^2.$$

We also know that:

$$\boxed{h_a = \frac{2r_b r_c}{r_b + r_c}}, \tag{10}$$

because both sides are equal to $\frac{2\Delta}{a}$; the analogous relations hold cyclically.

We put:

$$S_1 = r_a + r_b + r_c, \qquad S_2 = r_a r_b + r_b r_c + r_c r_a, \qquad S_3 = r_a r_b r_c.$$

From (10), after cyclic summation, we obtain:

$$\mathbf{\sum_{cyc} r_a\, h_a = 2S_3 \sum_{cyc} \frac{1}{S_1 - r_a}.} \tag{11}$$

The reciprocal sum can be written in the form:

$$\mathbf{\sum_{cyc} \frac{1}{S_1 - r_a} = \frac{S_1^2 + S_2}{S_1 S_2 - S_3}.} \tag{12}$$

Indeed, its numerator is:

$$(r_b + r_c)(r_c + r_a) + (r_c + r_a)(r_a + r_b) + (r_a + r_b)(r_b + r_c) = S_1^2 + S_2,$$

and its denominator is:

$$(r_a + r_b)(r_b + r_c)(r_c + r_a) = S_1 S_2 - S_3.$$

Using (7), (8) and (9) in (11) and (12), we obtain:

$$\mathbf{\sum_{cyc} r_a\, h_a = 2rs^2\frac{(4R + r)^2 + s^2}{(4R + r)s^2 - rs^2} = \frac{r((4R + r)^2 + s^2)}{2R}.} \tag{13}$$

From (6) and (13) we obtain the exact identity:

$$\boxed{\mathbf{n_a^2 + n_b^2 + n_c^2 = \frac{s^2(3R - r) - r(4R + r)^2}{R}.}} \tag{14}$$

For m = 2, using (5) and (14), inequality (1) becomes:

$$\frac{3}{r^2}\frac{s^2(3R - r) - r(4R + r)^2}{R} \le \frac{(4R + r)^2}{r^2}\frac{R}{2r}.$$

After multiplication by the positive quantity $2Rr^3$, we obtain the equivalent inequality:

$$6r(3R-r)s^2 \leq (4R+r)^2(R^2+6r^2).$$

Since $R \geq 2r$, we have $3R-r \geq 5r > 0$. Thus, it is sufficient to prove:

$$\boxed{s^2 \leq \frac{(4R+r)^2(R^2+6r^2)}{6r(3R-r)}.} \quad \textbf{(15)}$$

We now use the upper Blundon inequality:

$$\mathbf{s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}}. \quad \textbf{(16)}$$

Therefore, it remains to show that the right-hand side of (16) is not greater than the right-hand side of (15). We put:

$$x = \frac{R}{r} \geq 2, \qquad t = \sqrt{1-\frac{2}{x}} = \sqrt{\frac{R-2r}{R}}, \qquad 0 \leq t < 1.$$

Then:

$$x = \frac{2}{1-t^2}.$$

We consider the difference:

$$F(R,r) = \frac{(4R+r)^2(R^2+6r^2)}{6r(3R-r)} - \left[2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}\right].$$

After direct simplification, we obtain:

$$\mathbf{F(R,r)} = \frac{2r^2t^2(3-t)^2(12t^3+9t^2-6t+1)}{3(1-t)^3(1+t)^3(t^2+5)}. \quad \textbf{(17)}$$

Finally:

$$\mathbf{12t^3+9t^2-6t+1 = (1-3t)^2+12t^3 \geq 0}. \quad \textbf{(18)}$$

All the other factors in (17) are nonnegative, and the denominator is positive for $0 \leq t < 1$. Hence $F(R,r) \geq 0$. From (16), we obtain (15), and consequently inequality (1) is true for $m = 2$.

THE EQUALITY CASE.

In (17), the factor in (18) is strictly positive for $t > 0$, and $3-t > 0$. Therefore, equality is possible only for $t = 0$, that is:

$$t = 0 \Leftrightarrow R = 2r.$$

By Euler's inequality, $R = 2r$ holds only for the equilateral triangle.

Conversely, in the equilateral triangle:

$$R = 2r, \qquad n_a = n_b = n_c = 3r, \qquad q_a = q_b = q_c = 2r.$$

Therefore:

$$\frac{3(n_a^2+n_b^2+n_c^2)}{r^2} = 81,$$

and:

$$\left(\frac{n_a}{n_a-q_a}+\frac{n_b}{n_b-q_b}+\frac{n_c}{n_c-q_c}\right)^2\frac{R}{2r} = 9^2 = 81.$$

Thus, equality holds exactly for the equilateral triangle.

THE OPTIMALITY OF THE EXPONENT.

For an equilateral triangle, inequality (1) with an arbitrary integer exponent m would require:

$$81 \leq 9^m.$$

But, for every integer $m \leq 1$:

$$9^m \leq 9 < 81.$$

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Consequently, no integer exponent m ≤1 can satisfy (1) for all nondegenerate triangles. Since m = 2 has already been proved, we obtain:

$$\boxed{m_{opt} = 2}.$$

FINAL RESULT.

For every nondegenerate Euclidean triangle, we have:

$$\boxed{\frac{3\left(n_a^2+n_b^2+n_c^2\right)}{r^2} \le \left(\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}+\frac{n_b}{n_b-\sqrt{(c-a)^2+4r^2}}+\frac{n_c}{n_c-\sqrt{(a-b)^2+4r^2}}\right)^2\frac{R}{2r}}.$$

The exponent 2 is the least possible integer exponent, and equality holds only for the equilateral triangle.

A SHARP MAX-DIFFERENCE CONSEQUENCE.

For arbitrary real numbers x, y, z, define

$$M = \max\{(x-y)^2, (y-z)^2, (z-x)^2\}.$$

The following sharp algebraic inequality holds:

$$\mathbf{3(x^2+y^2+z^2) \ge (x+y+z)^2+\tfrac{3}{2}M.} \qquad \mathbf{(19)}$$

Indeed,

$$3(x^2+y^2+z^2)-(x+y+z)^2=(x-y)^2+(y-z)^2+(z-x)^2.$$

Assume, without loss of generality, that $M=(x-y)^2$. Since $x-y=(x-z)+(z-y)$, we have

$$(x-z)^2+(z-y)^2 \ge \frac{1}{2}((x-z)+(z-y))^2=\frac{1}{2}(x-y)^2.$$

Therefore, the preceding identity gives the stated inequality.

For the Nagel cevians, define

$$\mathbf{M_N=\max\{(n_a-n_b)^2, (n_b-n_c)^2, (n_c-n_a)^2\}.} \qquad \mathbf{(20)}$$

Applying the sharp algebraic inequality to $x=n_a$, $y=n_b$, and $z=n_c$, we obtain

$$\mathbf{3\left(n_a^2+n_b^2+n_c^2\right) \ge (n_a+n_b+n_c)^2+\tfrac{3}{2}M_N.} \qquad \mathbf{(21)}$$

Combining this estimate with the proved case $m=2$ of inequality (1), we obtain

$$\mathbf{\frac{(n_a+n_b+n_c)^2+\frac{3}{2}M_N}{r^2} \le \frac{3\sum_{cyc}n_a^2}{r^2} \le \left(\sum\frac{n_a}{n_a-q_a}\right)^2\frac{R}{2r}.} \qquad \mathbf{(22)}$$

Equality throughout this chain holds only for the equilateral triangle.

From (22) :

$$\mathbf{\frac{n_a+n_b+n_c}{r} \le \sqrt{\frac{R}{2r}}\sum\frac{n_a}{n_a-q_a}} \qquad \mathbf{(23)}$$

From (23):

$$\mathbf{n_a+n_b+n_c \le \sqrt{\frac{R}{2r}}(r_a+r_b+r_c)} \qquad \mathbf{(24)}$$

From (24) and (7):

$$\mathbf{n_a+n_b+n_c \le \sqrt{\frac{R}{2r}}(4R+r)} \qquad \mathbf{(25)}$$

From [2].(18): $\sqrt{\frac{R}{2r}} \ge \frac{s\sqrt{3}}{h_a+h_b+h_c}$ and (24) after summation: $\sqrt{\frac{R}{2r}}(r_a+r_b+r_c+h_a+h_b+h_c) \ge n_a+n_b+n_c+ s\sqrt{3}$

and multiplication:

$\frac{R}{2r} \geq \frac{s\sqrt{3}(n_a+n_b+n_c)}{(r_a+r_b+r_c)(h_a+h_b+h_c)}$ **(27)**

From [2].(29): $\sqrt{\frac{R}{2r}} \geq \frac{r_a+r_b+r_c}{m_a+m_b+m_c}$ **and (24) after summation:** $\sqrt{\frac{R}{2r}} \geq \frac{r_a+r_b+r_c+n_a+n_b+n_c}{m_a+m_b+m_c+r_a+r_b+r_c}$ **(28)**

and multiplication: $\frac{R}{2r} \geq \frac{n_a+n_b+n_c}{m_a+m_b+m_c}$ **(29)**

From (28) and $\sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3}}{h_a+h_b+h_c}$ **after summation:** $\sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3}+r_a+r_b+r_c+n_a+n_b+n_c}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c}$ **(30)**

From [2]. (26): $\sqrt{\frac{R}{2r}} \geq \frac{l_a}{h_a}$ **(and analogous) after summation:**

$\sqrt{\frac{R}{2r}}(h_a+h_b+h_c) \geq l_a+l_b+l_c \rightarrow \sqrt{\frac{R}{2r}} \geq \frac{l_a+l_b+l_c}{h_a+h_b+h_c}$ **(31)**

From (31) and $\sqrt{\frac{R}{2r}} \geq \frac{r_a+r_b+r_c}{m_a+m_b+m_c}$ **after summation:** $\sqrt{\frac{R}{2r}} \geq \frac{r_a+r_b+r_c+l_a+l_b+l_c}{h_a+h_b+h_c+m_a+m_b+m_c}$ **(32)**

From (24) and (32) after summation: $\sqrt{\frac{R}{2r}} \geq \frac{r_a+r_b+r_c+l_a+l_b+l_c+n_a+n_b+n_c}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c}$ **(33)**

From (30) and (33) after summation: $\sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3}+l_a+l_b+l_c+2(n_a+n_b+n_c+r_a+r_b+r_c)}{2(h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c)} \rightarrow$

$\sqrt{\frac{2R}{r}} \geq \frac{s\sqrt{3}+l_a+l_b+l_c+2(n_a+n_b+n_c+r_a+r_b+r_c)}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c}$ **(34)**

We know that : $s\sqrt{3} \geq m_a+l_b+l_c$ (and analogous)[3](§11.26(a), p. 221)and $\sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3}}{h_a+h_b+h_c}$ we obtain:

$\sqrt{\frac{R}{2r}} \geq \frac{m_a+l_b+l_c}{h_a+h_b+h_c}$ **(and analogous) (35)**

From (35) and $\sqrt{\frac{R}{2r}} \geq \frac{l_a}{h_a}$ (and analogous):

$\sqrt{\frac{R}{2r}} \geq \frac{m_a+l_a+l_b+l_c}{2h_a+h_b+h_c}$ **(and analogous)** **(36)**

From $\sqrt{\frac{R}{2r}} \geq \frac{m_b+l_a+l_c}{h_a+h_b+h_c}$ and $\sqrt{\frac{R}{2r}} \geq \frac{l_a}{h_a}$ obtain: $\sqrt{\frac{R}{2r}} \geq \frac{m_b+2l_a+l_c}{2h_a+h_b+h_c}$ (and analogous)
(37)

From (37) and (24) :

$\sqrt{\frac{R}{2r}} \geq \frac{m_b+2l_a+l_c+n_a+n_b+n_c}{2h_a+h_b+h_c+r_a+r_b+r_c}$ **(and analogous)** **(38)**

From (36) and (24):

$\sqrt{\frac{R}{2r}} \geq \frac{m_a+l_a+l_b+l_c+n_a+n_b+n_c}{2h_a+h_b+h_c+r_a+r_b+r_c}$ **(and analogous)** **(39)**

From (38) and (39):

$\sqrt{\frac{2R}{r}} \geq \frac{2(n_a+n_b+n_c)+m_a+m_b+3l_a+l_b+2l_c}{2h_a+h_b+h_c+r_a+r_b+r_c}$ **(and analogous)** **(40)**

From[5]: (28) :$\frac{b+c}{a} = \sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogous) and (24) we obtain:

$$\frac{b+c}{a}\geq\frac{2(n_a+n_b+n_c)}{r_a+r_b+r_c}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(and analogous) (41)}$$

From $2\Delta=ah_a=bh_b=ch_c=2sr=(a+b+c)r\to\frac{h_a}{r}=1+\frac{b+c}{a}\to\frac{b+c}{a}=\frac{h_a-r}{r}$ (and analogous) $\sum\frac{b+c}{a}=\frac{h_a+h_b+h_c-3r}{r}$. From (41) and $\frac{b+c}{a}=\frac{h_a-r}{r}$ (and analogous):

$$\frac{h_a-r}{2r}\geq\frac{n_a+n_b+n_c}{r_a+r_b+r_c}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(and analogous) (42)}$$

From (42) after summation : $\frac{h_a+h_b+h_c-3r}{2r}\geq\frac{n_a+n_b+n_c}{r_a+r_b+r_c}\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ **(43)**

From (42) : $\frac{l_a}{2r}\geq\frac{n_a+n_b+n_c}{r_a+r_b+r_c}\frac{h_a}{h_a-r}\sqrt{\frac{h_a}{r_a}}$ **(and analogous) (44)**

From (43) and $r_a+r_b+r_c=4R+r$: $\frac{1}{2}+\frac{2R}{r}\geq\frac{n_a+n_b+n_c}{h_a+h_b+h_c-3r}\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ **(45)**

From (44) after summation: $\frac{l_a+l_b+l_c}{2r}\geq\frac{n_a+n_b+n_c}{r_a+r_b+r_c}\sum\frac{h_a}{h_a-r}\sqrt{\frac{h_a}{r_a}}$ **(46)**

From (46) and $r_a+r_b+r_c=4R+r$: $\frac{1}{2}+\frac{2R}{r}\geq\frac{n_a+n_b+n_c}{l_a+l_b+l_c}\sum\frac{h_a}{h_a-r}\sqrt{\frac{h_a}{r_a}}$ **(47)**

From (45) and (47) after summation: $1+\frac{4R}{r}\geq\frac{n_a+n_b+n_c}{h_a+h_b+h_c-3r}\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}+\frac{n_a+n_b+n_c}{l_a+l_b+l_c}\sum\frac{h_a}{h_a-r}\sqrt{\frac{h_a}{r_a}}$ **(48)**

From (44): $\frac{l_al_bl_c}{8r^3}\geq\left(\frac{n_a+n_b+n_c}{r_a+r_b+r_c}\right)^3\sqrt{\frac{h_ah_bh_c}{r_ar_br_c}}\prod\frac{h_a}{h_a-r}$; Is well-known: $\frac{h_ah_bh_c}{r_ar_br_c}=\frac{2r}{R}$, we obtain:

$$\sqrt[6]{\frac{R}{2r}}\sqrt[3]{\frac{l_al_bl_c}{h_ah_bh_c}}\frac{\sqrt[3]{(h_a-r)(h_b-r)(h_c-r)}}{2r}\geq\frac{n_a+n_b+n_c}{r_a+r_b+r_c}\ \textbf{(49)}$$

From [6].(1): $n_a\geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous) and (49):

$$\sqrt[6]{\frac{R}{2r}}\sqrt[3]{\frac{l_al_bl_c}{h_ah_bh_c}}\frac{\sqrt[3]{(h_a-r)(h_b-r)(h_c-r)}}{r}\geq\frac{n_a+n_b+n_c+p_a\sqrt{\frac{l_a}{g_a}}+p_b\sqrt{\frac{l_b}{g_b}}+p_c\sqrt{\frac{l_c}{g_c}}}{r_a+r_b+r_c}\ \textbf{(50)}$$

From $n_a\geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous) and $n_a+n_b+n_c\leq\sqrt{\frac{R}{2r}}(r_a+r_b+r_c)$:

$$\sqrt{\frac{R}{2r}}(r_a+r_b+r_c)\geq p_a\sqrt{\frac{l_a}{g_a}}+n_b+n_c\ \textbf{(and analogous) (51)}$$

From [5]. (32): $\sum\frac{(r_a-r)}{l_a}\sqrt{\frac{h_a}{r_a}}=\sqrt{\frac{2R}{r}}$, $n_a+n_b+n_c\leq\sqrt{\frac{R}{2r}}(r_a+r_b+r_c)$ and $n_a\geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous):

$$\sum\frac{(r_a-r)}{l_a}\sqrt{\frac{h_a}{r_a}}\geq\frac{n_a+n_b+n_c+p_a\sqrt{\frac{l_a}{g_a}}+p_b\sqrt{\frac{l_b}{g_b}}+p_c\sqrt{\frac{l_c}{g_c}}}{r_a+r_b+r_c}\ \textbf{(52)}$$

From (5) , (45), (47) and (48):

$$\sum\frac{n_a}{n_a-q_a}\geq\frac{2(n_a+n_b+n_c)}{h_a+h_b+h_c-3r}\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\quad\textbf{(53)}$$

$$\sum\frac{n_a}{n_a-q_a}\geq\frac{2(n_a+n_b+n_c)}{l_a+l_b+l_c}\sum\frac{h_a}{h_a-r}\sqrt{\frac{h_a}{r_a}}\quad\textbf{(54)}$$

$$\sum\frac{n_a}{n_a-q_a}\geq\frac{n_a+n_b+n_c}{h_a+h_b+h_c-3r}\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}+\frac{n_a+n_b+n_c}{l_a+l_b+l_c}\sum\frac{h_a}{h_a-r}\sqrt{\frac{h_a}{r_a}}\ \textbf{(55)}$$

From [7]. :$n_a g_a \geq m_a l_a$ (and analogous) →$n_a \geq \frac{m_a l_a}{g_a}$ (and analogous) and (24) we obtain:

$$\sqrt{\frac{R}{2r}}(r_a + r_b + r_c) \geq \frac{m_a l_a}{g_a} + \frac{m_b l_b}{g_b} + \frac{m_c l_c}{g_c} \quad \textbf{(56)}$$

From (24) and (56):

$$\sqrt{\frac{2R}{r}} \geq \frac{n_a+n_b+n_c+\frac{m_a l_a}{g_a}+\frac{m_b l_b}{g_b}+\frac{m_c l_c}{g_c}}{r_a+r_b+r_c} \quad \textbf{(57)}$$

From (57) and $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous):

$$\sqrt{\frac{2R}{r}} \geq \frac{p_a\sqrt{\frac{l_a}{g_a}}+p_b\sqrt{\frac{l_b}{g_b}}+p_c\sqrt{\frac{l_c}{g_c}}+\frac{m_a l_a}{g_a}+\frac{m_b l_b}{g_b}+\frac{m_c l_c}{g_c}}{r_a+r_b+r_c} \quad \textbf{(58)}$$

From [8]. (4): $\frac{l_a}{h_a} \geq \frac{n_a-\sqrt{r^2+(b-c)^2}}{h_a-r}$ (and analogous) and $\sqrt{\frac{R}{2r}} \geq \frac{l_a}{h_a}$ (and analogous):

$$\sqrt{\frac{R}{2r}} \geq \frac{n_a-\sqrt{r^2+(b-c)^2}}{h_a-r} \quad \textbf{(and analogous) (59)}$$

From (59): $\sqrt{\frac{R}{2r}}(h_a - r) \geq n_a - \sqrt{r^2 + (b-c)^2}$ (and analogous) after summation:

$$\sqrt{\frac{R}{2r}}(h_a + h_b + h_c - 3r) \geq n_a + n_b + n_c - \sum\sqrt{r^2 + (b-c)^2} \quad \textbf{(60)}$$

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