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JP.630 In Imaginaryland there are 2027 towns. Every pair of towns is either connected by a single road, or is not connected. If we consider any subset of 2025 towns, the total number of roads connecting these towns to each other is a constant. If there are R roads in Imaginaryland, then find all possible values of R .

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Solution by proposer

We consider the general case with n towns. Let K denote the constant number of roads connecting any subset of $n - 2$ towns and let $c_{ij} \in \{0, 1\}$ denote the number roads connecting town i and town j . Finally, for $i = 1, 2, \dots, n$ let d_i denote the total number of roads connected to town i . Note that

$$R \leq \binom{n}{2} = \frac{n(n-1)}{2}$$

Clearly,

$$\sum_{i=1}^n d_i = 2R \quad \text{and} \quad \sum c_{ij} = R,$$

Where the latter sum is over all 2 – element subsets $\{i, j\}$ of $\{1, 2, \dots, n\}$. The number of roads connected to at least one of the towns with number i or j is equal to $d_i + d_j - c_{ij}$.

Thus, for any 2 – element subsets $\{i, j\} \subset \{1, 2, \dots, n\}$, we have

$$K = R - d_i - d_j + c_{ij}$$

Adding all these equations for every 2 – element subset $\{i, j\}$ yields

$$\binom{n}{2} K = \binom{n}{2} R - 2(n-1)R + R$$

which may be written as

$$n(n-1)K = (n-2)(n-3)R$$

Note that both $n(n-1)$ and $(n-2)(n-3)$ are divisible by 2 and that the only integer $k > 2$ which divides both $n(n-1)$ and $(n-2)(n-3)$ is 3, this latter case occurring if and only if n is divisible by 3. Since 3 does not divide 2027, in the situation of the given problem $\frac{n(n-1)}{2}$ and $\frac{(n-2)(n-3)}{2}$ are coprime. Hence, R is a multiple of $\frac{n(n-1)}{2}$. As $R \leq \frac{n(n-1)}{2}$

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with equality when all the pairs of towns are connected the only possibility is $R = \frac{n(n-1)}{2}$.

Therefore, the total number of roads is

$$R = 2027 \times 1013 = 2053351$$