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JP.629 The numbers $1, 2, \dots, 1000$ are written on the board. Two numbers are erased and replaced by the remainder of their sum when divided by 19. This operation is repeated until one number is left. What is the number?

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Solution by proposer

To find the final remaining number when the numbers $1, 2, \dots, 1000$ are repeatedly replaced by the remainder of their sum when divided by 19, we can use the concept of an invariant.

When two numbers, a and b , are erased and replaced by $(a + b)(\text{mod } 19)$, the change in the total sum of all numbers on the board is

$$\text{New Sum} = \text{Old Sum} - a - b + (a + b) (\text{mod } 19)$$

Because $(a + b) (\text{mod } 19)$ is congruent to $(a + b)$ modulo 19, the sum of all numbers on the board modulo 19 remains invariant at every single step. No matter which pairs are chosen or in what order, the final remaining single number will be congruent modulo 19 to the sum of the initial numbers.

The initial numbers form an arithmetic progression from 1 to 1000 which sum is

$$\text{Sum} = \frac{n(n+1)}{2} = \frac{1000 \times 1001}{2} = 500 \times 1001 = 500,500$$

Now, we compute $500,500 (\text{mod } 19)$ and we get

$$500,500 = 19 \times 26,342 + 2$$

Thus, the remainder when the total sum is divided by 19 is 2 and the final remaining number on the board is 2.