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JP.628 How many ways are there to put 15 balls into four bins so that the first two bins have between 2 and 6 balls, the third bin has at least 5 balls and the last bin has an odd number of balls?

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Solution by proposer

To solve the problem of finding the number of ways to distribute 15 identical balls into four bins subject to the given constraints, we model it as finding the number of integer solutions (x_1, x_2, x_3, x_4) to the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

subject to the constraints

1. $2 \leq x_1 \leq 6$
2. $2 \leq x_2 \leq 6$
3. $x_3 \geq 5$
4. $x_4 \geq 1$ and x_4 is odd

To convert these inequalities into non-negative integer variables, let

$$y_1 = x_1 - 2 \Rightarrow 0 \leq y_1 \leq 4$$

$$y_2 = x_2 - 2 \Rightarrow 0 \leq y_2 \leq 4$$

$$y_3 = x_3 - 5 \Rightarrow y_3 \geq 0$$

$$x_4 = 2y_4 + 1 \Rightarrow y_4 \geq 0 \text{ (since } x_4 \text{ is an odd positive integer)}$$

Substituting these into the original equation yields

$$(y_1 + 2) + (y_2 + 2) + (y_3 + 5) + (2y_4 + 1) = 15$$

or $y_1 + y_2 + y_3 + 2y_4 = 5$. Since y_4 is a non-negative integer, $2y_4$ can take the values 0, 2 or 4 (corresponding to $y_4 = 0, 1, 2$). We consider the following cases:

1. $y_4 = 0$ ($2y_4 = 0$). The equation becomes

$$y_1 + y_2 + y_3 = 5$$

with $0 \leq y_1 \leq 4$ and $0 \leq y_2 \leq 4$.

- Without upper bounds, the total number of non-negative integer solutions is

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$$\binom{5+3-1}{3-1} = \binom{7}{2} = 21$$

- We subtract invalid solutions where $y_1 \geq 5$ or $y_2 \geq 5$
- 1. If $y_1 \geq 5$, the only solution is $(y_1 = 5, y_2 = 0, y_3 = 0)$ (1 solution)
- 2. If $y_2 \geq 5$, the only solution is $(y_1 = 0, y_2 = 5, y_3 = 0)$ (1 solution)

Number of valid solutions for Case 1:

$$21 - 1 - 1 = 19$$

2. $y_4 = 1$ ($2y_4 = 2$). The equation becomes

$$y_1 + y_2 + y_3 = 3$$

with $0 \leq y_1 \leq 4$ and $0 \leq y_2 \leq 4$. All solutions naturally satisfy the upper bounds since the maximum value is 3.

$$\binom{3+3-1}{3-1} = \binom{5}{2} = 10$$

3. $y_4 = 2$ ($2y_4 = 4$). The equation becomes

$$y_1 + y_2 + y_3 = 1$$

with $0 \leq y_1 \leq 4$ and $0 \leq y_2 \leq 4$. All solutions naturally satisfy the upper bounds.

$$\binom{1+3-1}{3-1} = \binom{3}{2} = 3$$

Finally, summing the valid solutions from all three cases, we obtain

$$\text{Total ways} = 19 + 10 + 3 = 32$$