

ROMANIAN MATHEMATICAL MAGAZINE

JP.627 If $a, b, c > 0$ and $\lambda \geq 0$ then:

$$\sum \frac{\sqrt{a^2 + (b + \lambda c)^2}}{b + \lambda c} \geq \frac{6\lambda + 9}{(\lambda + 1)\sqrt{5}}$$

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Solution by proposer

We prove: Lemma:

If $a, b, c > 0$ and $\lambda \geq 0$ then:

$$\frac{\sqrt{a^2 + (b + 2\lambda c)^2}}{b + \lambda c} \geq \frac{1}{\sqrt{5}} \left(2 + \frac{a}{b + \lambda c} \right)$$

Solution:

$$\frac{\sqrt{a^2 + (b + \lambda c)^2}}{b + \lambda c} = \sqrt{\frac{a^2}{(b + \lambda c)^2} + 1} = \sqrt{t^2 + 1} \stackrel{(1)}{\geq} \frac{1}{\sqrt{5}}(t + 2), \text{ where } (1) \Leftrightarrow \sqrt{t^2 + 1} \geq \frac{1}{\sqrt{5}}(t + 2) \Leftrightarrow$$

$$\Leftrightarrow 5(t^2 + 1) \geq (t + 2)^2 \Leftrightarrow (2t - 1)^2 \geq 0, \text{ obviously with equality for } t = \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{a}{b + \lambda c} = \frac{1}{2} \Leftrightarrow b + \lambda c = 2a$$

Let's get back to the main problem. Using the Lemma, we obtain:

$$\begin{aligned} LHS &= \sum \frac{\sqrt{a^2 + (b + \lambda c)^2}}{b + \lambda c} \stackrel{\text{Lemma}}{\geq} \sum \frac{1}{\sqrt{5}} \left(2 + \frac{a}{b + \lambda c} \right) = \frac{1}{\sqrt{5}} \left(6 + \sum \frac{a}{b + \lambda c} \right) \stackrel{\text{Nesbitt}}{\geq} \\ &\geq \frac{1}{\sqrt{5}} \left(6 + \frac{3}{\lambda + 1} \right) = \frac{1}{\sqrt{5}} \left(\frac{6\lambda + 9}{\lambda + 1} \right) = \frac{6\lambda + 9}{(\lambda + 1)\sqrt{5}} = RHS \end{aligned}$$

Equality holds if and only if $a = b = c = \lambda = 1$.