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JP.626 If $a, b, c > 0, a^n + b^n + c^n = 3$ and $n \in \mathbb{N}^*$ then:

$$\sum \frac{a}{a^{2n+1} + b^{2n} + c^{2n-1}} \leq 1$$

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Solution by proposer

We prove: Lemma:

If $a, b, c > 0, a^n + b^n + c^n = 3$ then:

$$\frac{a}{a^{2n+1} + b^{2n} + c^{2n-1}} \leq \frac{1 + a + ac}{9}$$

Using CBS inequality we obtain:

$$\begin{aligned} (a^{2n+1} + b^{2n} + c^{2n-1}) \left(\frac{1}{a} + 1 + c \right) &\geq (a^n + b^n + c^n)^2 = 9 \Leftrightarrow \\ \Leftrightarrow \frac{a}{a^{2n+1} + b^{2n} + c^{2n-1}} &\leq \frac{1 + a + ac}{9}. \end{aligned}$$

Let's get back to the main problem. Using the Lemma, we obtain:

$$\begin{aligned} LHS &= \sum \frac{a}{a^{2n+1} + b^{2n} + c^{2n-1}} \stackrel{\text{Lemma}}{\leq} \sum \frac{1 + a + ac}{9} = \frac{1}{9} \left(3 + \sum a + \sum ac \right) \stackrel{(1)}{\leq} \\ &\stackrel{(1)}{\leq} \frac{1}{9} (3 + 3 + 3) = 1 = RHS, \end{aligned}$$

where (1) $\Leftrightarrow \sum ac \leq 3$, which follows from $\sum bc \leq \frac{1}{3} (\sum a)^2 \stackrel{(2)}{\leq} \frac{1}{3} (3)^2 = 3$,

where (2) $\Leftrightarrow \sum a \leq 3$, which follows from Holder's inequality:

$$\sum a^n \stackrel{\text{Holder}}{\geq} \frac{(\sum a)^n}{3^{n-1}}, \sum a^n = 3 \Rightarrow 3 \stackrel{\text{Holder}}{\geq} \frac{(\sum a)^n}{3^{n-1}} \Rightarrow \sum a \leq 3$$

Equality holds if and only if $a = b = c = 1$.