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JP.625 If $a, b, c \geq 0, a + b + c = 2$ and $\lambda \geq 1$ then:

$$\sum \sqrt{\frac{b+c}{a^2+\lambda}} \geq \frac{4}{\sqrt{\lambda+1}}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

We prove: Lemma:

If $0 \leq x \leq 2$ and $\lambda \geq 1$ then:

$$\sqrt{\frac{2-x}{x^2+\lambda}} \geq \frac{2-x}{\sqrt{1+\lambda}}$$

Proof:

We have $\sqrt{\frac{2-x}{x^2+\lambda}} \geq \frac{2-x}{\sqrt{1+\lambda}} \Leftrightarrow x(2-x)(x-1)(x^2-x+\lambda-1) \geq 0$, which follows from $\lambda \geq$

1 with equality for $x \in \{0, 1, 2\}$. Let's get back to the main problem:

Using the Lemma we obtain:

$$\begin{aligned} LHS &= \sum \sqrt{\frac{b+c}{a^2+\lambda}} = \sum \sqrt{\frac{2-a}{a^2+\lambda}} \stackrel{\text{Lemma}}{\geq} \sum \frac{2-a}{\sqrt{\lambda+1}} = 3 \cdot \frac{2}{\sqrt{\lambda+1}} - \frac{1}{\sqrt{\lambda+1}} \sum a = \\ &= 3 \cdot \frac{2}{\sqrt{\lambda+1}} - \frac{1}{\sqrt{\lambda+1}} \cdot 2 = \frac{4}{\sqrt{\lambda+1}} = RHS \end{aligned}$$