

ROMANIAN MATHEMATICAL MAGAZINE

JP.624 If $a, b, c > 0, a + b + c = 3$ and $0 \leq \lambda \leq 2$ then:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{\lambda}{abc} \leq 1$$

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Solution by proposer

$$\begin{aligned} \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{\lambda}{abc} &= \frac{1}{3} \sum \frac{1}{a^2} + \frac{2}{3} \sum \frac{1}{a^2} - \frac{\lambda}{abc} \cdot \frac{a+b+c}{3} = \\ &= \frac{1}{3} \left[\sum \left(\frac{1}{a^2} + a + a \right) - 2 \sum a \right] + \frac{2-\lambda}{3} \sum \frac{1}{a^2} + \frac{\lambda}{3} \left(\sum \frac{1}{a^2} - \sum \frac{1}{bc} \right) \stackrel{AGM}{\geq} \\ &\stackrel{AGM}{\geq} \frac{1}{3} \left[\sum 3 \sqrt[3]{\frac{1}{a^2} \cdot a \cdot a} - 2 \cdot 3 \right] + \frac{2-\lambda}{3} \sum \frac{1}{a^2} + \frac{\lambda}{3} \left(\sum \frac{1}{a^2} - \sum \frac{1}{bc} \right) = \\ &= \frac{1}{3} \left(\sum 3 \sqrt[3]{1} - 2 \cdot 3 \right) + \frac{2-\lambda}{3} \sum \frac{1}{a^2} + \frac{\lambda}{6} \sum \left(\frac{1}{b} - \frac{1}{c} \right)^2 \stackrel{0 \leq \lambda \leq 2}{=} \frac{1}{3} (9 - 6) + 0 \cdot \sum \frac{1}{a^2} + 0 \\ &= 1. \end{aligned}$$

Equality holds if and only if $a = b = c = 1$ and $\lambda = 2$.