

# ROMANIAN MATHEMATICAL MAGAZINE

**JP.623** Find all values of  $n$  in the range  $1 \leq n \leq 10$  for which the sum

$$S = \sum_{k=1}^{n^2+n} [\sqrt{k}]$$

is a prime number. Here  $[x]$  represents the integer part of  $x$ .

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**Solution by proposer**

Let  $M = n^2 + n$ . We evaluate the sum by grouping terms according to the value of  $j = [\sqrt{k}]$ . The condition  $[\sqrt{k}] = j$  is equivalent to

$$j^2 \leq k < (j+1)^2$$

Since  $M = n^2 + n$  satisfies  $n^2 \leq n^2 + n < (n+1)^2$ , the maximum integer value for  $j$  is  $n$ . For each  $j$  from 1 to  $n-1$ , the number of terms is  $(j+1)^2 - j^2 = 2j+1$ . For  $j = n$ , the terms run from  $n^2$  to  $n^2 + n$ , giving a total of  $(n^2 + n) - n^2 + 1 = n+1$  terms. Thus, the sum can be expressed as

$$\begin{aligned} S &= \sum_{j=1}^{n-1} j(2j+1) + n(n+1) = \sum_{j=1}^{n-1} (2j^2 + j) + n(n+1) \\ &= 2 \left( \frac{(n-1)n(2n-1)}{6} \right) + \frac{(n-1)n}{2} + n(n+1) \end{aligned}$$

from which we get the closed form polynomial.

$$S = \frac{n(4n^2 + 3n + 5)}{6}$$

Substituting integers  $n$  from 1 to 10 into the expression yields

$$\text{For } n = 1: \frac{1(4(1)^2 + 3(1) + 5)}{6} = 2 \text{ (Prime)}$$

$$\text{For } n = 2: \frac{2(4(2)^2 + 3(2) + 5)}{6} = 9 \text{ (Composite)}$$

$$\text{For } n = 3: \frac{3(4(3)^2 + 3(3) + 5)}{6} = 25 \text{ (Composite)}$$

$$\text{For } n = 4: \frac{4(4(4)^2 + 3(4) + 5)}{6} = 54 \text{ (Composite)}$$

$$\text{For } n = 5: \frac{5(4(5)^2 + 3(5) + 5)}{6} = 100 \text{ (Composite)}$$

$$\text{For } n = 6: \frac{6(4(6)^2 + 3(6) + 5)}{6} = 167 \text{ (Prime)}$$

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For  $n = 7$ :  $\frac{7(4(7)^2+3(7)+5)}{6} = 259$  (Composite)

For  $n = 8$ :  $\frac{8(4(8)^2+3(8)+5)}{6} = 380$  (Composite)

For  $n = 9$ :  $\frac{9(4(9)^2+3(9)+5)}{6} = 534$  (Composite)

For  $n = 10$ :  $\frac{10(4(10)^2+3(10)+5)}{6} = 725$  (Composite)

Finally, we conclude that the values of  $n$  in the range  $1 \leq n \leq 10$  for which the sum is a prime number are  $n = 1$  and  $n = 6$ .