ROMANIAN MATHEMATICAL MAGAZINE

JP. 622 Let a, b, c be positive reals such that $a+b+c=3$. Find the minimum value of

$$\frac{a^2-ab+b^2}{b+c}+\frac{b^2-bc+c^2}{c+a}+\frac{c^2-ca+a^2}{a+b}.$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

Since the given expression equals to $\frac{3}{2}$ when $a=b=c=1$ then we conjecture that

$$\frac{a^2-ab+b^2}{b+c}+\frac{b^2-bc+c^2}{c+a}+\frac{c^2-ca+a^2}{a+b}\geq\frac{3}{2}.$$

Indeed, for any real numbers x and y, we have the identity

$$x^2-xy+y^2=\frac{3}{4}(x-y)^2+\frac{1}{4}(x+y)^2\geq\frac{1}{4}(x+y)^2$$

Applying this bound to each numerator in the given expression gives

$$\sum\frac{a^2-ab+b^2}{b+c}\geq\frac{1}{4}\left(\frac{(a+b)^2}{b+c}+\frac{(b+c)^2}{c+a}+\frac{(c+a)^2}{a+b}\right)$$

By the Cauchy – Schwarz inequality, we have

$$\sum_{cyc}\frac{(a+b)^2}{b+c}\geq\frac{\big((a+b)+(b+c)+(c+a)\big)^2}{(b+c)+(c+a)+(a+b)}=\frac{(2a+2b+2c)^2}{2(a+b+c)}=2(a+b+c)$$

Substituting this back into our inequality yields

$$\sum_{cyc}\frac{a^2-ab+b^2}{b+c}\geq\frac{1}{4}\cdot2(a+b+c)=\frac{a+b+c}{2}$$

Since $a+b+c=3$, then it follows that

$$\sum_{cyc}\frac{a^2-ab+b^2}{b+c}\geq\frac{3}{2}$$

Solution 2 by Marin Chirciu-Romania

$$P=\sum\frac{a^2-ab+b^2}{b+c}\overset{SOS}{\geq}\sum\frac{a^2+b^2}{2(b+c)}\overset{CS}{\geq}\frac{1}{2}\cdot\frac{(\sum a)^2+(\sum b)^2}{\sum(b+c)}=\frac{1}{2}\cdot\frac{2(\sum a)^2}{2\sum a}=\frac{1}{2}\sum a=\frac{3}{2}$$

Equality holds if and only if $a=b=c=1$.

We deduce that $\min P=\frac{3}{2}$ for $(a,b,c)=(1,1,1)$.

Remark: The problem can be developed.

If $a,b,c>0,\sum a=3$ and $\lambda\le 2$ then find min of

$$\sum\frac{a^2-\lambda ab+b^2}{b+c}$$

Marin Chirciu

Solution:

$$P=\sum\frac{a^2-\lambda ab+b^2}{b+c}\overset{sos}{\ge}\sum\frac{(2-\lambda)(a^2+b^2)}{2(b+c)}\overset{cs}{\ge}\frac{2-\lambda}{2}\cdot\frac{(\sum a)^2+(\sum b)^2}{\sum(b+c)}=$$

$$=\frac{2-\lambda}{2}\cdot\frac{2(\sum a)^2}{2\sum a}=\frac{2-\lambda}{2}\cdot\sum a=\frac{3}{2}(2-\lambda).$$

Equality holds if and only if $a=b=c=1$.

We deduce that $\min P=\frac{3}{2}(2-\lambda)$ for $(a,b,c)=(1,1,1)$.

Note: For $\lambda=1$ we obtain Problem JP.622 from RMM Number 42 – Autumn 2026.

JP.622. If $a,b,c>0,\sum a=3$ then find min of

$$\sum\frac{a^2-ab+b^2}{b+c}$$

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