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JP.621 Determine all prime numbers p, q, r such that:

$$p^2 + q^2 - r(p + q) = 108$$

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Solution by proposer

Considering the equation modulo 2, we get

$$p^2 + q^2 - r(p + q) \equiv 108 \pmod{2}$$

Since $x^2 \equiv x \pmod{2}$ and $108 \equiv 0 \pmod{2}$, this simplifies to

$$(p + q) - r(p + q) \equiv 0 \pmod{2}$$

If one of the primes is 2 (let's say $p = 2$ and q is an odd prime), then $p - q$ and $p + q$ are both odd. The equation becomes:

$$\text{Odd} - r(\text{Odd}) \equiv 0 \pmod{2}$$

This requires r to be an odd prime.

Next, we analyze the cases where one of the variables is a small prime. We distinguish the following cases:

1. $p = 2$ and q is an odd prime. Substituting $p = 2$ into the original equation yields

$2^2 + q^2 - r(2 + q) = 108$ from which we get

$$r = \frac{104 - q^2}{q - 2} = -(q + 2) + \frac{100}{q - 2}$$

For r to be an integer, $q - 2$ must be a positive divisor of 100.

The divisors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, 100.

Testing $q - 2 = 5$, which gives $q = 7$ (a prime number)

$$r = -(7 + 2) + \frac{100}{5} = -9 + 20 = 11$$

Since 11 is a prime number, this yields a valid solution: $(p, q, r) = (2, 7, 11)$

2. Explicit check of other divisors. For completeness, we test the remaining positive divisors of 100 for $q - 2$, we have

- $q - 2 = 1 \Rightarrow q = 3, r = -5 + 100 = 95$ (Not prime)
- $q - 2 = 2 \Rightarrow q = 4$ (Not prime)

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- $q - 2 = 4 \Rightarrow q = 6$ (Not prime)
- $q - 2 = 10 \Rightarrow q = 12$ (Not prime)
- $q - 2 = 20 \Rightarrow q = 22$ (Not prime)
- $q - 2 = 25 \Rightarrow q = 27$ (Not prime)
- $q - 2 = 50 \Rightarrow q = 52$ (Not prime)
- $q - 2 = 100 \Rightarrow q = 102$ (Not prime)

3. Checking $q = 2$ with p as an Odd Prime. If we swap roles and set $q = 2$, the formula for r becomes

$$r = (p + 2) - \frac{100}{p - 2}$$

Testing the divisors of 100 for $p - 2$ yields

- $p - 2 = 1 \Rightarrow p = 3, r = 5 - 100 = -95$ (Negative, invalid)
- $p - 2 = 5 \Rightarrow p = 7, r = 9 - 20 = -11$ (Negative, invalid)
- All other divisors yields either non-prime values for p or negative values for r .

4. Checking $p = q$. If $p = q$, the term $r(p - q)$ vanishes:

$$p^2 + p^2 = 108 \Rightarrow 2p^2 = 108 \Rightarrow p^2 = 54$$

Since 54 is not a square, no solutions exist in this case.

Thus, the only set of prime numbers (p, q, r) satisfying the equation is

$$(p, q, r) = (2, 7, 11)$$