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JP.620 Determine the number of positive integers a such that $a|12!$ but 18 does not divide a .

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Solution by proposer

The prime factorization of $12!$ involves all prime numbers less than or equal to 12, which are $\{2, 3, 5, 7, 11\}$. Using Legendre's formula, the exponents are

$$\text{For } p = 2: \left\lfloor \frac{12}{2} \right\rfloor + \left\lfloor \frac{12}{4} \right\rfloor + \left\lfloor \frac{12}{8} \right\rfloor = 6 + 3 + 1 = 10 \Rightarrow 2^{10}$$

$$\text{For } p = 3: \left\lfloor \frac{12}{3} \right\rfloor + \left\lfloor \frac{12}{9} \right\rfloor = 4 + 1 = 5 \Rightarrow 3^5$$

$$\text{For } p = 5: \left\lfloor \frac{12}{5} \right\rfloor = 2 \Rightarrow 5^2$$

$$\text{For } p = 7: \left\lfloor \frac{12}{7} \right\rfloor = 1 \Rightarrow 7^1$$

$$\text{For } p = 11: \left\lfloor \frac{12}{11} \right\rfloor = 1 \Rightarrow 11^1,$$

and

$$12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1$$

Next, the prime factorization of 18 is $18 = 2^1 \cdot 3^2$. Any divisor a of $12!$ takes the general form

$$a = 2^{e_2} \cdot 3^{e_3} \cdot 5^{e_5} \cdot 7^{e_7} \cdot 11^{e_{11}}$$

where the exponents are bounded by:

- $0 \leq e_2 \leq 10$
- $0 \leq e_3 \leq 5$
- $0 \leq e_5 \leq 2$
- $0 \leq e_7 \leq 1$
- $0 \leq e_{11} \leq 1$

The condition states that 18 does not divide a ($18 \nmid a$). Since $18 = 2^1 \cdot 3^2$, for 18 to divide a , we would need to have at least one factor of 2 and at least two factors of 3.

Therefore, 18 does divide a if and only if $e_2 \geq 1$ and $e_3 \geq 2$. Consequently, 18 does not divide a if and only if this condition fails, meaning

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$$e_2 = 0 \quad \text{or} \quad e_3 < 2$$

To find the number of valid integers a , it is easier to find the total number of divisors of $12!$ and subtract the ones that are divisible by 18. We have

1. Total divisors of $12!$:

$$\text{Total} = (10 + 1)(5 + 1)(2 + 1)(1 + 1)(1 + 1) = 11 \times 6 \times 3 \times 2 \times 2 = 792$$

2. Divisors of $12!$ That are divisible by 18: For $18|a$, the exponents must satisfy $1 \leq e_2 \leq 10$ and $2 \leq e_3 \leq 5$ (while e_5, e_7, e_{11} have their normal ranges).

- Choices for e_2 : 10 choices (1 to 10)
- Choices for e_3 : 4 choices (2 to 5)
- Choices for e_5 : 3 choices (0 to 2)
- Choices for e_7 : 2 choices (0 to 1)
- Choices for e_{11} : 2 choices (0 to 1)

$$\text{Divisors divisible by 18} = 10 \times 4 \times 3 \times 2 \times 2 = 480$$

Finally, subtracting the invalid divisors from the total yields

$$\text{Valid divisors} = 792 - 480 = 312$$