ROMANIAN MATHEMATICAL MAGAZINE

JP.619 Let $a_1, a_2, \dots, a_n$ be nonzero real numbers such that

$a_1 + a_2 + a_n = 1$. Find the minimum value of

$$\sum_{k=1}^{n} \frac{a_k^3}{a_k^2 + a_k a_{k+1} + a_{k+1}^2}$$

where the subscripts are taken modulo n.

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

First observe that when $a_1 = a_2 = \dots = a_n = \frac{1}{n}$ the given sum equals $\frac{1}{3}$. So, we conjecture that

$$\sum_{k=1}^{n} \frac{a_k^3}{a_k^2 + a_k a_{k+1} + a_{k+1}^2} \geq \frac{1}{3}.$$

Indeed, on account of the constrain, instead of the preceding we will prove the stronger inequality

$$\sum_{k=1}^{n} \frac{a_k^3}{a_k^2 + a_k a_{k+1} a_{k+1}^2} \geq \frac{a_1 + a_2 + \dots + a_n}{3}.$$

To do it, we consider the identity

$$\frac{a_1^3 - a_2^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_2^3 - a_3^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_n^3 - a_1^3}{a_n^2 + a_n a_1 + a_1^2}$$
$$= (a_1 - a_2) + (a_2 - a_3) + \dots + (a_n - a_1) = 0,$$

from which it follows

$$\frac{a_1^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_2^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_n^3}{a_n^2 + a_n a_1 + a_1^2}$$
$$= \frac{a_2^3}{a_1^2 + a_1 a_2} + \frac{a_3^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_1^3}{a_n^2 + a_n a_1 + a_1^2}$$

Now, we have

$$2\left(\frac{a_1^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_2^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_n^3}{a_n^2 + a_n a_1 + a_1^2}\right)$$

$$=\frac{a_1^3+a_2^3}{a_1^2+a_1a_2+a_2^2}+\frac{a_2^3+a_3^3}{a_2^2+a_2a_3+a_3^2}+\cdots+\frac{a_n^3+a_1^3}{a_n^2+a_na_1+a_1^2}$$

$$=\frac{(a_1+a_2)(a_1^2-a_1a_2+a_2^2)}{a_1^2+a_1a_2+a_2^2}+\cdots+\frac{(a_n+a_1)(a_n^2-a_na_1+a_1^2)}{a_n^2+a_na_1+a_n^2}$$

$$\geq\frac{1}{3}[(a_1+a_2)+(a_2+a_3)+\cdots+(a_n+a_1)]=\frac{2}{3}(a_1+a_2+\cdots+a_n)$$

The last inequality is true on account that

$$\frac{a_k^2-a_ka_{k+1}+a_{k+1}^2}{a_k^2+a_ka_{k+1}+a_{k+1}^2}\geq\frac{1}{3}$$

as can be easily checked. Then, we have:

$$\sum_{k=1}^{n}\frac{a_k^3}{a_k^2+a_ka_{k+1}+a_{k+1}^2}\geq\frac{1}{3},$$

with equality when $a_1=a_2=\cdots=a_n=\frac{1}{n}$, and the conjecture is proven.

Finally, we may conclude that the minimum value of the given expression is $\frac{1}{3}$, and we are done.

Solution 2 by Marin Chirciu-Romania

Using $\frac{x^3}{x^2+xy+y^2}\geq\frac{2x-y}{3}\Leftrightarrow(x-y)^2(x+y)\geq 0$ we obtain:

$$P=\sum_{k=1}^{n}\frac{a_k^3}{a_k^2+a_ka_{k+1}+a_{k+1}^2}=\sum_{k=1}^{n}\frac{2a_k-a_{k+1}}{3}=\frac{1}{3}\sum_{k=1}^{n}a_k=\frac{1}{3}\cdot 1=\frac{1}{3}$$

Equality holds if and only if $a_1=a_2=\cdots=a_n=\frac{1}{n}$.

We deduce that $\min P=\frac{1}{3}$ for $a_1=a_2=\cdots=a_n=\frac{1}{n}$.

Remark: The problem can be developed.

If $a_1,a_2,\dots,a_n\geq 0,\sum_{k=1}^{n}a_k=1$ and $\lambda\geq 0$ then find min of

$$\sum_{k=1}^{n}\frac{a_k^3}{a_k^2+\lambda a_ka_{k+1}+a_{k+1}^2}$$

Marin Chirciu

Solution:

Using $\frac{x^3}{x^2+\lambda xy+y^2} \geq \frac{2x-y}{\lambda+2} \Leftrightarrow (x-y)^2(\lambda x+y) \geq 0$ **we obtain:**

$$P = \sum_{k=1}^{n} \frac{a_k^3}{a_k^2+\lambda a_k a_{k+1}+a_{k+1}^2} = \sum_{k=1}^{n} \frac{2a_k - a_{k+1}}{\lambda+2} = \frac{1}{\lambda+2}\sum_{k=1}^{n} a_k = \frac{1}{\lambda+2}\cdot 1 = \frac{1}{\lambda+2}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$.

We deduce that $\min P = \frac{1}{\lambda+2}$ **for** $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$.

Note: For $\lambda = 1$ **we obtain Problem JP.619 from RMM Number 42 – Autumn 2026.**

JP.619. If $a_1, a_2, \dots, a_n \geq 0, \sum_{k=1}^{n} a_k = 1$ **then find min of**

$$\sum_{k=1}^{n} \frac{a_k^3}{a_k^2+a_k a_{k+1}+a_{k+1}^2}$$

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