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JP. 618 Let $x_1, x_2, \dots, x_m > 0$, where $m \geq 2$, and use the cyclic convention $x_{m+1} = x_1$. If $\alpha > 1$ and $\lambda \geq 1$, prove that

$$\sum_{k=1}^m \frac{x_k((\alpha - 1)x_k + 2x_{k+1})}{\alpha x_k^2 + x_{k+1}^2} + \lambda \sum_{k=1}^m \frac{(x_k - x_{k+1})^2}{x_k^2 + x_{k+1}^2} \geq m,$$

and determine all equality cases.

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Solution by proposer

The inequality follows from a termwise estimate. Fix an index k and write

$$x = x_k, \quad y = x_{k+1}.$$

Since $x, y > 0$, all denominators below are positive. First observe the following exact identity:

$$\frac{x((\alpha - 1)x + 2y)}{\alpha x^2 + y^2} + \frac{(x - y)^2}{x^2 + y^2} - 1 = \frac{(\alpha - 1)x^2(x - y)^2}{(\alpha x^2 + y^2)(x^2 + y^2)}.$$

Indeed, after putting the left-hand side over the common denominator

$(\alpha x^2 + y^2)(x^2 + y^2)$, its numerator is

$$\begin{aligned} & x((\alpha - 1)x + 2y)(x^2 + y^2) + (x - y)^2(\alpha x^2 + y^2) \\ & - (\alpha x^2 + y^2)(x^2 + y^2) = (\alpha - 1)x^2(x - y)^2. \end{aligned}$$

We now restore the coefficient λ . Splitting $\lambda = 1 + (\lambda - 1)$ in the second fraction give

$$\begin{aligned} \frac{x((\alpha - 1)x + 2y)}{\alpha x^2 + y^2} + \lambda \frac{(x - y)^2}{x^2 + y^2} - 1 &= \frac{(\alpha - 1)x^2(x - y)^2}{(\alpha x^2 + y^2)(x^2 + y^2)} + (\alpha - 1) \frac{(x - y)^2}{x^2 + y^2} = \\ &= \frac{(x - y)^2}{x^2 + y^2} \left(\lambda - 1 + \frac{(\alpha - 1)x^2}{\alpha x^2 + y^2} \right). \end{aligned}$$

The last expression is nonnegative because $\lambda - 1 \geq 0$, $\alpha - 1 > 0$, and $x, y > 0$.

Consequently, for every $k = 1, \dots, m$

$$\frac{x_k((\alpha - 1)x_k + 2x_{k+1})}{\alpha x_k^2 + x_{k+1}^2} + \lambda \frac{(x_k - x_{k+1})^2}{x_k^2 + x_{k+1}^2} \geq 1.$$

Adding these m inequalities proves the required lower bound m .

Equality

Because $\alpha > 1$ and $x, y > 0$, the coefficient

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$$\lambda - 1 + \frac{(\alpha - 1)x^2}{\alpha x^2 + y^2}$$

is strictly positive. Hence equality in the k – th term occurs if and only if

$$x_k = x_{k+1}.$$

Equality in the sum therefore holds if and only if is true for every index k . With $x_{m+1} = x_1$, this is equivalent to

$$x_1 = x_2 = \cdots = x_m > 0.$$

Conversely, if all the variables are equal, each first fraction is 1 and every squared-difference fraction is 0, so the left-hand side is exactly m . Thus the displayed family gives all equality cases.