

ROMANIAN MATHEMATICAL MAGAZINE

JP. 617 Let a, b, c, d be nonnegative real numbers such that

$a \geq b \geq c \geq d$ and $(a + d)(b + c) = 4$. Prove that:

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} \geq 1.$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Denoting

$$x = \frac{a+d}{2}, \quad y = \frac{b+c}{2},$$

we have $xy = 1$. By the AM-HM inequality, we get

$$\frac{1}{b+3} + \frac{1}{c+3} \geq \frac{2}{y+3}.$$

So, it suffices to show that

$$\frac{1}{a+3} + \frac{1}{d+3} + \frac{2}{y+3} \geq 1,$$

i.e.

$$\frac{1}{a+3} + \frac{1}{d+3} \geq \frac{y+1}{y+3}, \quad \frac{2(x+3)}{ad+6x+9} \geq \frac{y+1}{y+3}.$$

From $(y-a)(y-d) \leq 0$, we get $ad \leq 2xy - y^2 = 2 - y^2$, hence

$$\frac{2(x+3)}{ad+6x+9} \geq \frac{2(x+3)}{(2-y^2)+6x+9} = \frac{2(3y+1)}{-y^3+11y+6}$$

So, it suffices to show that

$$\frac{2(3y+1)}{-y^3+11y+6} \geq \frac{y+1}{y+3},$$

which is equivalent to

$$y(y^3 + y^2 - 5y + 3) \geq 0, \quad y(y+3)(y-1)^2 \geq 0.$$

The equality occurs for $a = b = c = d = 1$.