

ROMANIAN MATHEMATICAL MAGAZINE

JP.616 Determine the largest positive value of the constant k such that:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} - 4 \geq k(a + b + c + d - 4)$$

for any positive real numbers a, b, c, d satisfying $ab + bc + cd + da = 4$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a = c = x > 0$ and $b = d = \frac{1}{x}$, the constraint is satisfied and the inequality becomes

$$x^2 + \frac{1}{x^2} - 2 \geq k\left(x + \frac{1}{x} - 2\right), \quad (x - 1)^2[(x - 1)^2 + 4 - k] \geq 0.$$

For $x \rightarrow 1$, we get the necessary condition $k \leq 4$. We claim that 4 is the largest positive value of k . To show this, we need to prove that the inequality holds for $k = 4$, that is

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + 12 \geq 4(a + b + c + d).$$

Let

$$x = \frac{a + c}{2}, \quad y = \frac{b + d}{2}, \quad xy = 1.$$

By the AM-GM inequality, we have

$$\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{ac} \geq \frac{8}{(a + c)^2} = \frac{2}{x^2}.$$

Similarly,

$$\frac{1}{b^2} + \frac{1}{d^2} \geq \frac{2}{y^2}.$$

So, it suffices to show that

$$\frac{1}{x^2} + \frac{1}{y^2} + 6 \geq 4(x + y), \quad \frac{1}{x^2} + x^2 + 6 \geq 4\left(x + \frac{1}{x}\right),$$

$$\left(x + \frac{1}{x}\right)^2 + 4 \geq 4\left(x + \frac{1}{x}\right), \quad \left(x + \frac{1}{x} - 2\right)^2 \geq 0.$$

If $k = 4$, then the equality occurs when $a = b = c = d = 1$.