Generalizations, Sharp Remainder Refinements, and Companion Inequalities of Huygens Type

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Abstract

We give a self-contained correction, generalization, and sharpening of a family of Huygens-type inequalities. Weighted circular and hyperbolic Huygens inequalities are characterized completely. The circular Huygens and Mitrinović–Adamović inequalities are strengthened by convergent hierarchies whose coefficients are successively best possible. We also derive a sharp logarithmic-variance refinement of weighted AM–GM, an exact remainder version of an auxiliary sum inequality, and a complete positive remainder decomposition of the original refinement chain. Further companion results include a sharp functional remainder for the hyperbolic Huygens inequality, an exact exponential remainder for the Lazarević inequality, optimal power exponents, and refined circular and hyperbolic Wilker inequalities. All equality and sharpness assertions are proved explicitly.

Keywords: Huygens inequality; Mitrinović–Adamović inequality; Lazarević inequality; Wilker inequality; trigonometric functions; hyperbolic functions; weighted inequality; sharp constant; exact remainder.

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1 Introduction

For $0 < x < \pi/2$, the classical Huygens inequality is

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > 3, \tag{1}$$

or, equivalently,

$$2\sin x + \tan x > 3x. \tag{2}$$

The Mitrinović–Adamović inequality is

$$\frac{\sin x}{x} > (\cos x)^{1/3}. \tag{3}$$

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These inequalities, their hyperbolic counterparts, and their Wilker-type relatives have been studied in, among other sources, [5, 6, 4, 7, 3, 8, 2, 9].

The source manuscript proposed the chain

$$\begin{aligned} 2\sin x + \tan x &\geq \sin x \left(1 + 2\sqrt[4]{\frac{1+\cos^2 x}{2\cos^3 x}}\right) \\ &\geq \sin x \left(1 + \frac{2}{\sqrt{\cos x}}\right) \geq \frac{3\sin x}{\sqrt[3]{\cos x}} \geq 3x. \end{aligned} \tag{4}$$

The chain is valid after correcting a typographical error in its proof. However, its auxiliary n-variable lemma is not valid as printed. We first repair and strengthen the algebraic tools, and then derive substantially stronger trigonometric conclusions. No claim of historical priority is intended for identities that follow directly from classical product expansions; the purpose is to give one complete, verifiable treatment of all inequalities used in the source.

The additional results below are organized around a circular–hyperbolic duality. In the circular weighted Huygens inequality the admissible weights satisfy $\alpha \leq 2\beta$, whereas in the hyperbolic companion they satisfy the reversed condition $\alpha \geq 2\beta$. Exact identities then transfer the Huygens refinements to Wilker-type inequalities without losing the positive remainder terms.

Throughout, $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ denotes the Riemann zeta function.

2 Sharpened algebraic tools

We begin with a weighted refinement of AM–GM. It contains the usual n-variable AM–GM inequality and quantifies its gap.

Theorem 2.1 (weighted AM–GM with a sharp remainder)**.** *Let $a_1, \dots, a_n > 0$ and let $w_1, \dots, w_n > 0$ satisfy $\sum_{i=1}^{n} w_i = 1$. Put*

$$A_w = \sum_{i=1}^{n} w_i a_i, \qquad G_w = \prod_{i=1}^{n} a_i^{w_i}, \qquad m = \min_{1\leq i\leq n} a_i.$$

Then

$$A_w - G_w \geq \frac{m}{2} \sum_{i=1}^{n} w_i \big(\log a_i - \log G_w\big)^2 \geq 0. \tag{5}$$

Equality holds if and only if $a_1 = \cdots = a_n$. The numerical coefficient $1/2$ in (5) cannot be increased uniformly.

Proof. Set $y_i = \log a_i$ and $\bar{y} = \sum_i w_i y_i = \log G_w$. Every point between y_i and $\bar{y}$ is at least $\log m$. Taylor's theorem with integral remainder, applied to $f(y) = \mathrm{e}^y$, therefore gives

$$\mathrm{e}^{y_i} \geq \mathrm{e}^{\bar{y}} + \mathrm{e}^{\bar{y}}(y_i - \bar{y}) + \frac{m}{2}(y_i - \bar{y})^2.$$

Multiplication by w_i and summation cancel the linear terms and prove (5). Equality in the first inequality forces $y_i = \bar{y}$ for every i, hence all the a_i are equal.

To prove sharpness, take $n = 2$, $w_1 = w_2 = 1/2$, $a_1 = \mathrm{e}^{\varepsilon}$, and $a_2 = \mathrm{e}^{-\varepsilon}$. Then $G_w = 1$, $m = \mathrm{e}^{-\varepsilon}$, and

$$\frac{A_w - G_w}{m \sum_i w_i (\log a_i - \log G_w)^2} = \frac{\cosh \varepsilon - 1}{\mathrm{e}^{-\varepsilon} \varepsilon^2} \longrightarrow \frac{1}{2} \qquad (\varepsilon \to 0^+).$$

Thus no larger universal coefficient is possible. □

The next identity repairs the auxiliary sum inequality used in the source.

Theorem 2.2 (corrected sum inequality and exact remainder)**.** *Let $a_i, b_i > 0$ for $i = 1, \ldots, n$, and define*

$$u_i = \sqrt{a_i b_i}, \qquad v_i = \sqrt{\frac{a_i^2 + b_i^2}{2}}, \qquad \bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i, \qquad \bar{v} = \frac{1}{n} \sum_{i=1}^{n} v_i.$$

Then the exact identity

$$\sum_{i=1}^{n} (a_i + b_i)^2 = \frac{4}{n} \left(\sum_{i=1}^{n} u_i \right) \left(\sum_{i=1}^{n} v_i \right) + 2n(\bar{u} - \bar{v})^2 + 2 \sum_{i=1}^{n} (u_i - \bar{u})^2 + 2 \sum_{i=1}^{n} (v_i - \bar{v})^2 \tag{6}$$

holds. Consequently,

$$\sum_{i=1}^{n} (a_i + b_i)^2 \geq \frac{4}{n} \left(\sum_{i=1}^{n} \sqrt{a_i b_i} \right) \left(\sum_{i=1}^{n} \sqrt{\frac{a_i^2 + b_i^2}{2}} \right). \tag{7}$$

The constant $4/n$ is best possible. Equality holds precisely when $a_1 = b_1 = \cdots = a_n = b_n$.

Proof. For every i,

$$(a_i + b_i)^2 = a_i^2 + b_i^2 + 2a_i b_i = 2(u_i^2 + v_i^2).$$

Also,

$$\sum_i u_i^2 = n\bar{u}^2 + \sum_i (u_i - \bar{u})^2, \qquad \sum_i v_i^2 = n\bar{v}^2 + \sum_i (v_i - \bar{v})^2.$$

Substituting these formulas and using

$$2n(\bar{u}^2 + \bar{v}^2) = 4n\bar{u}\bar{v} + 2n(\bar{u} - \bar{v})^2$$

proves (6); dropping the three nonnegative remainders gives (7).

Equality requires $u_i = \bar{u}$, $v_i = \bar{v}$ for all i, and $\bar{u} = \bar{v}$. Since $u_i = v_i$ is equivalent to $(a_i - b_i)^2 = 0$, all a_i and b_i must have one common value. Conversely, that condition clearly gives equality. Finally, if all $a_i = b_i = a$, then both sides of (7) are equal. Hence $4/n$ cannot be replaced by any larger constant. □

Remark 2.3 (the error in the source lemma). The factor $1/n$ is missing from the displayed lemma in the source manuscript. Without it, take $n = 2$ and $a_i = b_i = 1$. The asserted inequality would be $8 \geq 16$, which is false. The proof of the trigonometric chain uses only the case $n = 1$, for which the missing factor happens to be invisible. For $n = 1$, (6) becomes

$$(a + b)^2 = 4uv + 2(u - v)^2, \quad u = \sqrt{ab}, \quad v = \sqrt{\frac{a^2 + b^2}{2}}. \tag{8}$$

Taking square roots and rationalizing gives the exact refinement

$$a+b=2\sqrt{uv}+\frac{2(u-v)^2}{a+b+2\sqrt{uv}}. \tag{9}$$

3 An all-order sharpening of Huygens' inequality

We use the Euler product

$$\cos x=\prod_{k=0}^{\infty}\left(1-\frac{4x^2}{(2k+1)^2\pi^2}\right), \qquad |x|<\frac{\pi}{2}.$$

Taking logarithms and differentiating term by term yields

$$\tan x=\sum_{n=1}^{\infty}\frac{2(4^n-1)\zeta(2n)}{\pi^{2n}}x^{2n-1}, \qquad |x|<\frac{\pi}{2}. \tag{10}$$

Theorem 3.1 (sharp Huygens hierarchy)**.** *For $n\geq 3$, define*

$$h_n=\frac{2(4^n-1)\zeta(2n)}{\pi^{2n}}+\frac{2(-1)^{n-1}}{(2n-1)!}. \tag{11}$$

Then $h_n>0$, and for $0<|x|<\pi/2$,

$$\frac{2\sin x}{x}+\frac{\tan x}{x}=3+\sum_{n=3}^{\infty}h_n x^{2n-2}. \tag{12}$$

Consequently, for every integer $N\geq 3$,

$$\frac{2\sin x}{x}+\frac{\tan x}{x}>3+\sum_{n=3}^{N}h_n x^{2n-2}. \tag{13}$$

Every coefficient h_N is best possible once the preceding coefficients $h_3,\dots,h_{N-1}$ have been fixed.

Proof. Combine (10) with the power series of $2\sin x$. The coefficient of x is 3, while the coefficient of x^3 vanishes. For $n\geq 3$, the coefficient of x^{2n-1} is exactly h_n, proving (12).

It remains to check positivity. If n is odd, both terms in (11) are positive. If $n\geq 4$ is even, it is enough, since $\zeta(2n)>1$, to prove

$$(4^n-1)(2n-1)!>\pi^{2n}.$$

Using $\pi^2<10$, the assertion follows at $n=4$ from $(4^4-1)7!>10^4$, and then follows for all larger n by induction because the ratio of consecutive left-hand sides is greater than $4(2n)(2n+1)>10$. Thus all h_n are positive, so discarding the positive tail proves (13).

Finally, after the terms through x^{2N-4} have been fixed, division of the remaining difference by x^{2N-2} and passage to $x\to 0$ gives h_N. No larger coefficient can therefore hold in a lower bound near the origin. □

The first three nonzero coefficients are

$$h_3 = \frac{3}{20}, \qquad h_4 = \frac{3}{56}, \qquad h_5 = \frac{7}{320}.$$

Thus the following directly sharpens (1).

Corollary 3.2. *For* $0 < |x| < \pi/2$,

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > 3 + \frac{3x^4}{20} + \frac{3x^6}{56} + \frac{7x^8}{320}. \tag{14}$$

In particular, $3/20$ *is the largest constant* C *for which*

$$\frac{2\sin x}{x} + \frac{\tan x}{x} \geq 3 + Cx^4$$

holds throughout $0 < |x| < \pi/2$.

We now characterize the complete two-weight generalization.

Theorem 3.3 (optimal weighted Huygens inequality)**.** *Let* $\alpha \geq 0$ *and* $\beta > 0$. *The inequality*

$$\alpha \sin x + \beta \tan x \geq (\alpha + \beta)x \qquad \left(0 < x < \frac{\pi}{2}\right) \tag{15}$$

holds for every x *in the indicated interval if and only if*

$$\alpha \leq 2\beta. \tag{16}$$

If (16) *holds, then for every* $N \geq 3$ *the stronger inequality*

$$\begin{aligned} \alpha \sin x + \beta \tan x - (\alpha + \beta)x > (2\beta - \alpha)(x - \sin x) \\ + \beta \sum_{n=3}^{N} h_n x^{2n-1} \end{aligned} \tag{17}$$

holds.

Proof. The exact identity

$$\begin{aligned} &\alpha \sin x + \beta \tan x - (\alpha + \beta)x \\ &\quad = \beta(2\sin x + \tan x - 3x) + (2\beta - \alpha)(x - \sin x) \end{aligned} \tag{18}$$

follows by expansion. Under (16), both terms on the right are nonnegative, and Theorem 3.1 gives (17), hence (15).

Conversely, as $x \to 0^+$,

$$\alpha \sin x + \beta \tan x - (\alpha + \beta)x = \frac{2\beta - \alpha}{6}x^3 + O(x^5).$$

If $\alpha > 2\beta$, the expression is negative for all sufficiently small positive x. Therefore (16) is necessary. □

Taking $(\alpha, \beta) = (2, 1)$ recovers Huygens' inequality together with all the refinements in Theorem 3.1.

4 A sharp hierarchy for the Mitrinović–Adamović inequality

The next result both generalizes the exponent in (3) and adds an arbitrarily long sequence of sharp correction terms.

Theorem 4.1 (exact logarithmic expansion)**.** *For $n \geq 2$, set*

$$d_n = \frac{(4^n - 4)\zeta(2n)}{3n\pi^{2n}} > 0, \qquad P_N(x) = \sum_{n=2}^{N} d_n x^{2n}. \tag{19}$$

Then, for $0 < |x| < \pi/2$,

$$\log\left(\frac{\sin x}{x(\cos x)^{1/3}}\right) = \sum_{n=2}^{\infty} d_n x^{2n}. \tag{20}$$

Hence, for every $N \geq 2$,

$$\frac{\sin x}{x} > (\cos x)^{1/3} \exp\bigl(P_N(x)\bigr). \tag{21}$$

Every d_N is best possible successively after $d_2, \ldots, d_{N-1}$ have been fixed.

Proof. Euler's products give, for $|x| < \pi/2$,

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right), \qquad \cos x = \prod_{k=0}^{\infty}\left(1 - \frac{4x^2}{(2k+1)^2\pi^2}\right).$$

Taking logarithms, expanding $\log(1-y)$, and summing absolutely on compact subintervals gives

$$\log\frac{\sin x}{x} = -\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n\pi^{2n}} x^{2n},$$
$$\log\cos x = -\sum_{n=1}^{\infty} \frac{(4^n - 1)\zeta(2n)}{n\pi^{2n}} x^{2n}.$$

Subtracting one third of the second identity from the first cancels the x^2 term and gives (20). Since every d_n is positive, discarding the positive tail and exponentiating proves (21). The same limiting argument as in Theorem 3.1 proves successive sharpness. □

Since

$$d_2 = \frac{1}{45}, \qquad d_3 = \frac{4}{567}, \qquad d_4 = \frac{1}{450},$$

we obtain the explicit refinement

$$\frac{\sin x}{x} > (\cos x)^{1/3} \exp\left(\frac{x^4}{45} + \frac{4x^6}{567} + \frac{x^8}{450}\right). \tag{22}$$

Corollary 4.2 (optimal power exponent)**.** *Let $p \in \mathbb{R}$. Then*

$$\frac{\sin x}{x} \geq (\cos x)^p \qquad \left(0 < x < \frac{\pi}{2}\right) \tag{23}$$

holds for every x if and only if $p \geq 1/3$. More strongly, if $p \geq 1/3$ and $N \geq 2$, then

$$\frac{\sin x}{x} > (\cos x)^p \exp\bigl(P_N(x)\bigr). \tag{24}$$

Proof. If $p \geq 1/3$, then $0 < \cos x < 1$ implies $(\cos x)^p \leq (\cos x)^{1/3}$, so (24) follows from Theorem 4.1. Conversely, expansion at the origin gives

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + O(x^4), \qquad (\cos x)^p = 1 - \frac{p}{2}x^2 + O(x^4).$$

Thus (23) near zero requires $p \geq 1/3$. □

5 A fully sharpened form of the source chain

We now replace every weak step in (4) by an exact positive remainder. For $0 < x < \pi/2$, write

$$s = \sin x, \qquad c = \cos x, \qquad t = \tan x, \qquad A(c) = \sqrt[4]{\frac{1+c^2}{2c^3}}.$$

Define

$$u = \sqrt{st}, \qquad\qquad v = \sqrt{\frac{s^2+t^2}{2}}, \tag{25}$$

$$\mathcal{R}(x) = \frac{2(u-v)^2}{s+t+2sA(c)}, \tag{26}$$

$$\Delta(c) = A(c) - c^{-1/2}, \tag{27}$$

$$\mathcal{E}(c) = \big(c^{-1/6}-1\big)^2\big(2c^{-1/6}+1\big). \tag{28}$$

Theorem 5.1 (exact decomposition and sharpened chain)**.** *For $0 < x < \pi/2$, all three quantities $\mathcal{R}(x)$, $\Delta(c)$, and $\mathcal{E}(c)$ are positive, and*

$$2\sin x + \tan x = \frac{3\sin x}{(\cos x)^{1/3}} + \mathcal{R}(x) + 2\sin x\,\Delta(\cos x) + \sin x\,\mathcal{E}(\cos x). \tag{29}$$

Moreover, for every integer $N \geq 2$,

$$\begin{aligned} 2\sin x + \tan x &> 3x\exp\big(P_N(x)\big) + \mathcal{R}(x) + 2\sin x\,\Delta(\cos x) \\ &\quad + \sin x\,\mathcal{E}(\cos x) > 3x. \end{aligned} \tag{30}$$

In particular, this is a strict refinement of every link in (4).

Proof. First apply the exact two-variable formula (9) with $a = s$ and $b = t$. A direct calculation gives

$$\sqrt{uv} = \sqrt[4]{\frac{st(s^2+t^2)}{2}} = s\sqrt[4]{\frac{1+c^2}{2c^3}} = sA(c).$$

Therefore

$$2s + t = s + (s+t) = s\big(1+2A(c)\big) + \mathcal{R}(x). \tag{31}$$

This proves the first link of the source chain and supplies its exact omitted remainder.

Next, rationalization of fourth powers gives

$$\Delta(c) = \frac{(1-c)^2}{2c^3\big(A(c)^3 + A(c)^2c^{-1/2} + A(c)c^{-1} + c^{-3/2}\big)} > 0. \tag{32}$$

Hence

$$s\big(1+2A(c)\big) = s\big(1+2c^{-1/2}\big) + 2s\Delta(c). \tag{33}$$

This is the exact strengthened form of the second link.

For the third link, set $q = c^{-1/6} > 1$. The elementary factorization

$$1 + 2q^3 - 3q^2 = (q-1)^2(2q+1) \tag{34}$$

gives

$$1 + 2c^{-1/2} = 3c^{-1/3} + \mathcal{E}(c). \tag{35}$$

Notice that the corresponding line in the source proof must read $1+2/\sqrt{c} \geq 3/\sqrt[3]{c}$ before multiplication by $\sin x$; the factor $\sin x$ appearing there prematurely is a typographical error.

Combining (31), (33), and (35) proves (29). Theorem 4.1 gives

$$\frac{s}{c^{1/3}} > x\exp\big(P_N(x)\big).$$

Substitution in (29) proves the first inequality in (30). Its second inequality follows because $P_N(x) > 0$ and all three remainders are positive. □

The first member of the hierarchy already gives the concise explicit bound

$$2\sin x + \tan x > 3x\exp\left(\frac{x^4}{45}\right) + \mathcal{R}(x) + 2\sin x\,\Delta(\cos x) + \sin x\,\mathcal{E}(\cos x). \tag{36}$$

One may alternatively use the purely polynomial hierarchy

$$2\sin x + \tan x > 3x + \frac{3x^5}{20} + \frac{3x^7}{56} + \frac{7x^9}{320}, \tag{37}$$

which follows from Corollary 3.2. The constants in (37) are successively best possible at the origin.

6 Companion hyperbolic and Wilker-type inequalities

The preceding circular inequalities admit useful hyperbolic companions. We give elementary proofs with explicit remainders; thus the results in this section do not depend on analytic continuation of the circular formulas.

6.1 A sharp functional remainder for hyperbolic Huygens

For $y \geq 1$, define

$$\Phi(y) = \frac{2y+1}{y^2}. \tag{38}$$

The function Φ is strictly decreasing because $\Phi'(y) = -2(y+1)/y^3 < 0$.

Theorem 6.1 (hyperbolic Huygens with an optimal functional remainder)**.** *For every $x > 0$,*

$$2\sinh x + \tanh x - 3x = \int_0^x \frac{(\cosh t - 1)^2(2\cosh t + 1)}{\cosh^2 t}\,\mathrm{d}t \tag{39}$$

$$> \frac{x^5}{20}\Phi(\cosh x) > 0. \tag{40}$$

The coefficient $1/20$ in (40) is best possible among constants multiplying $x^5\Phi(\cosh x)$.

Proof. Let

$$H_{mathrmh}(x) = 2\sinh x + \tanh x - 3x.$$

Since $H_{\mathrm{h}}(0) = 0$, differentiation and the substitution $y = \cosh x$ give

$$\begin{aligned} H_{\mathrm{h}}'(x) &= 2\cosh x + \frac{1}{\cosh^2 x} - 3 \\ &= \frac{2y^3 - 3y^2 + 1}{y^2} = \frac{(y-1)^2(2y+1)}{y^2}. \end{aligned}$$

Integration proves (39).

For $0 \le t \le x$, the Taylor series of $\cosh t$ gives $\cosh t - 1 \ge t^2/2$, with strict inequality when $t > 0$. Moreover, $\Phi(\cosh t) \ge \Phi(\cosh x)$ because Φ is decreasing. Therefore

$$\begin{aligned} H_{\mathrm{h}}(x) &= \int_0^x (\cosh t - 1)^2\Phi(\cosh t)\,\mathrm{d}t \\ &> \Phi(\cosh x)\int_0^x \frac{t^4}{4}\,\mathrm{d}t = \frac{x^5}{20}\Phi(\cosh x). \end{aligned}$$

Finally,

$$H_{\mathrm{h}}(x) = \frac{3x^5}{20} + O(x^7), \qquad \Phi(\cosh x) = 3 + O(x^2) \quad (x \to 0^+).$$

Hence

$$\lim_{x\to 0^+} \frac{H_{\mathrm{h}}(x)}{x^5\Phi(\cosh x)} = \frac{1}{20},$$

which proves optimality. □

The weighted hyperbolic analogue of Theorem 3.3 has the reverse weight condition.

Theorem 6.2 (optimal weighted hyperbolic Huygens inequality)**.** *Let $\alpha \ge 0$ and $\beta > 0$. Then*

$$\alpha\sinh x + \beta\tanh x \ge (\alpha + \beta)x \qquad (x > 0) \tag{41}$$

holds for every $x > 0$ if and only if $\alpha \ge 2\beta$. Under this condition, the stronger estimate

$$\begin{aligned} &\alpha\sinh x + \beta\tanh x - (\alpha+\beta)x \\ &\qquad > \frac{\beta x^5}{20}\Phi(\cosh x) + (\alpha - 2\beta)(\sinh x - x) \end{aligned} \tag{42}$$

holds for all $x > 0$.

Proof. The identity

$$\begin{aligned}\alpha \sinh x + \beta \tanh x - (\alpha+\beta)x \\ = \beta(2\sinh x + \tanh x - 3x) + (\alpha - 2\beta)(\sinh x - x)\end{aligned} \tag{43}$$

is immediate. If $\alpha \geq 2\beta$, both terms on the right are nonnegative, and Theorem 6.1 gives (42).

Conversely, expansion at the origin yields

$$\alpha \sinh x + \beta \tanh x - (\alpha+\beta)x = \frac{\alpha - 2\beta}{6}x^3 + O(x^5).$$

When $\alpha < 2\beta$, this expression is negative for all sufficiently small positive x. Thus $\alpha \geq 2\beta$ is necessary. □

6.2 An exact Lazarević remainder and the optimal exponent

The hyperbolic companion of the Mitrinović–Adamović inequality is usually called the Lazarević inequality. The following form records its entire positive logarithmic gap.

Theorem 6.3 (exact exponential Lazarević refinement)**.** *For $x > 0$, put*

$$G(x) = 3x + 2x\sinh^2 x - 3\sinh x \cosh x, \tag{44}$$

$$\Lambda(x) = \int_0^x \frac{G(t)}{3t \sinh t \cosh t}\,\mathrm{d}t, \tag{45}$$

where the integrand is understood by continuity at $t = 0$. Then $\Lambda(x) > 0$ and

$$\frac{\sinh x}{x} = (\cosh x)^{1/3} \exp\bigl(\Lambda(x)\bigr) > (\cosh x)^{1/3}. \tag{46}$$

Furthermore,

$$\Lambda(x) = \frac{x^4}{45} - \frac{4x^6}{567} + O(x^8) \qquad (x \to 0^+). \tag{47}$$

Proof. Define

$$L(x) = \log \frac{\sinh x}{x} - \frac{1}{3}\log\cosh x.$$

A common denominator gives

$$L'(x) = \coth x - \frac{1}{x} - \frac{1}{3}\tanh x = \frac{G(x)}{3x \sinh x \cosh x}. \tag{48}$$

To determine the sign of G, set

$$J(x) = x\cosh x - \sinh x.$$

Then $J(0) = 0$ and $J'(x) = x\sinh x > 0$, so $J(x) > 0$ for $x > 0$. Direct differentiation of (44) now yields

$$G'(x) = 4\sinh x\, J(x) > 0.$$

As $G(0) = 0$, we have $G(x) > 0$, and hence $L'(x) > 0$. Since $L(0^+) = 0$, integration of (48) gives $L(x) = \Lambda(x) > 0$. Exponentiation proves (46). Finally, the Taylor series of sinh and cosh in the definition of L gives (47). □

Corollary 6.4 (optimal hyperbolic power exponent)**.** *For* $p \in \mathbb{R}$*, the inequality*

$$\frac{\sinh x}{x} \geq (\cosh x)^p \qquad (x > 0) \tag{49}$$

holds for every $x > 0$ *if and only if* $p \leq 1/3$*.*

Proof. If $p \leq 1/3$, then $\cosh x > 1$ and $(\cosh x)^p \leq (\cosh x)^{1/3} < \sinh x/x$ by Theorem 6.3. Conversely,

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{6} + O(x^4), \qquad (\cosh x)^p = 1 + \frac{p}{2}x^2 + O(x^4).$$

Thus validity near the origin requires $p \leq 1/3$. □

Corollaries 4.2 and 6.4 display the duality clearly: the strongest circular power bound requires $p \geq 1/3$, whereas the strongest hyperbolic power bound requires $p \leq 1/3$.

6.3 Refined Wilker-type consequences

The Huygens refinements imply Wilker-type inequalities through an exact algebraic relation.

Theorem 6.5 (circular Wilker hierarchy)**.** *For* $0 < |x| < \pi/2$ *and every integer* $N \geq 3$*,*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \left(\frac{\sin x}{x} - 1\right)^2 + \sum_{n=3}^{N} h_n x^{2n-2}. \tag{50}$$

In particular,

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{79x^8}{3600}. \tag{51}$$

The coefficient $8/45$ *is the largest constant* C *for which*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} \geq 2 + Cx^4$$

holds on $0 < |x| < \pi/2$*.*

Proof. For $a = \sin x/x$ and $b = \tan x/x$, the exact identity

$$a^2 + b - 2 = (2a + b - 3) + (a - 1)^2 \tag{52}$$

holds. Theorem 3.1 immediately gives (50).

For $0 < |x| < \pi/2$, the alternating Taylor estimate gives

$$\frac{\sin x}{x} < 1 - \frac{x^2}{6} + \frac{x^4}{120}.$$

The quantity $x^2/6 - x^4/120$ is positive on this interval; hence

$$\left(1 - \frac{\sin x}{x}\right)^2 > \left(\frac{x^2}{6} - \frac{x^4}{120}\right)^2.$$

Use this inequality and the terms h_3, h_4, h_5 in (50). Collecting coefficients gives

$$\frac{3}{20}+\frac{1}{36}=\frac{8}{45}, \qquad \frac{3}{56}-\frac{1}{360}=\frac{16}{315}, \qquad \frac{7}{320}+\frac{1}{14400}=\frac{79}{3600},$$

which proves (51). Finally, direct expansion gives

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2=\frac{8x^4}{45}+O(x^6).$$

Therefore no larger leading constant is possible. $\square$

Theorem 6.6 (hyperbolic Wilker refinement)**.** *For every $x>0$,*

$$\begin{aligned}\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x} &> 2+\left(\frac{\sinh x}{x}-1\right)^2+\frac{x^4}{20}\Phi(\cosh x)\\ &> 2+x^4\left(\frac{1}{36}+\frac{2\cosh x+1}{20\cosh^2 x}\right).\end{aligned} \tag{53}$$

The final functional bound is asymptotically sharp at the origin.

Proof. Apply the identity (52) with $a=\sinh x/x$ and $b=\tanh x/x$. The first inequality follows from Theorem 6.1 after division by x. Since

$$\frac{\sinh x}{x}>1+\frac{x^2}{6},$$

we have $(\sinh x/x-1)^2>x^4/36$, which proves the second inequality. Moreover, as $x\to 0^+$,

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}-2=\frac{8x^4}{45}+O(x^6),$$

while the coefficient in parentheses on the last line of (53) tends to $1/36+3/20=8/45$. This proves asymptotic sharpness. $\square$

7 Concluding remarks

The original Huygens and Mitrinović–Adamović bounds are therefore not isolated estimates. They are the first members of convergent positive-term hierarchies. In the circular and hyperbolic weighted Huygens problems the optimal conditions are, respectively, $\alpha \le 2\beta$ and $\alpha \ge 2\beta$. The circular and hyperbolic power exponents are likewise dual: $p \ge 1/3$ and $p \le 1/3$. The corrected factor $4/n$ in the auxiliary sum inequality is optimal, while the original main chain decomposes exactly into its terminal Mitrinović–Adamović term and three explicit positive remainders. The exact Huygens–Wilker identity then preserves these remainders in both the circular and hyperbolic settings. Thus every stated inequality is accompanied by either an exact gap formula or a proved sharp constant, and no unverified step remains in the argument.

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