ROMANIAN MATHEMATICAL MAGAZINE

CALCULATION OF ASSOCIATED TRIANGLE AREAS

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1. Notation and Refinement of the Classical Triangle Inequality

Let ABC be a nondegenerate Euclidean triangle with side lengths $\mathrm{a}, \mathrm{b}, \mathrm{c}$, semiperimeter

$\mathrm{s}=\frac{\mathrm{a}+\mathrm{b}+\mathrm{c}}{2}$, area Δ, inradius r, and circumradius R. Define

$$\mathrm{A_r}=\sqrt{(\mathrm{b}-\mathrm{c})^2+4\mathrm{r}^2},\qquad \mathrm{B_r}=\sqrt{(\mathrm{c}-\mathrm{a})^2+4\mathrm{r}^2},\qquad \mathrm{C_r}=\sqrt{(\mathrm{a}-\mathrm{b})^2+4\mathrm{r}^2}.\qquad (1)$$

1.1. Lower refined thresholds: independent Heron-form proof

Heron's formula gives: $\mathrm{a}^2-\mathrm{A_r^2}=\mathrm{a}^2-(\mathrm{b}-\mathrm{c})^2-4\mathrm{r}^2=\frac{4\mathrm{a}(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})}{\mathrm{s}}>0.$

Since $\mathrm{a}>0$ and $\mathrm{A_r}>0$, it follows that: $\mathrm{A_r}<a$.

By cyclic permutation: $\mathrm{B_r}<b, \qquad \mathrm{C_r}<c$.

Moreover, because $\mathrm{r}>0$, $\mathrm{A_r}=\sqrt{(\mathrm{b}-\mathrm{c})^2+4\mathrm{r}^2}>|\mathrm{b}-\mathrm{c}|$,

and cyclically: $\mathrm{B_r}>|\mathrm{c}-\mathrm{a}|, \qquad \mathrm{C_r}>|\mathrm{a}-\mathrm{b}|$.

Therefore: $|\mathrm{b}-\mathrm{c}|<\mathrm{A_r}<a, |\mathrm{c}-\mathrm{a}|<\mathrm{B_r}<b, |\mathrm{a}-\mathrm{b}|<\mathrm{C_r}<c$.

Thus, $\mathrm{A_r}, \mathrm{B_r}, \mathrm{C_r}$ are strict inradius-dependent lower thresholds for the corresponding original sides.

1.2. Existence of the associated derived triangle

The quantities $\mathrm{A_r}, \mathrm{B_r}, \mathrm{C_r}$ satisfy the strict triangle inequalities. Let

$$\mathbf{u}=(\mathrm{b}-\mathrm{c}, 2\mathrm{r}),\qquad \mathbf{v}=(\mathrm{c}-\mathrm{a}, 2\mathrm{r}).\ \text{Then: } \|\mathbf{u}\|=\mathrm{A_r},\qquad \|\mathbf{v}\|=\mathrm{B_r}.$$

By the triangle inequality for the Euclidean norm in $\mathbb{R}^2$,

$$\mathrm{A_r}+\mathrm{B_r}\geq\|\mathbf{u}+\mathbf{v}\|=\sqrt{(\mathrm{b}-\mathrm{a})^2+16\mathrm{r}^2}.$$

Because $\mathrm{r}>0$, $\sqrt{(\mathrm{b}-\mathrm{a})^2+16\mathrm{r}^2}>\sqrt{(\mathrm{a}-\mathrm{b})^2+4\mathrm{r}^2}=\mathrm{C_r}$.

Hence: $\mathrm{A_r}+\mathrm{B_r}>\mathrm{C_r}$. Cyclic relabeling gives

$$B_r + C_r > A_r, \qquad C_r + A_r > B_r.$$

Therefore, (A_r, B_r, C_r) is the side-length triple of a nondegenerate Euclidean triangle. Scaling by the positive factor $1/(2r)$ also shows that

$$\frac{A_r}{2r}, \qquad \frac{B_r}{2r}, \qquad \frac{C_r}{2r}$$

are the side lengths of a nondegenerate triangle.

1.3. Cross-derived upper inequalities

For every nondegenerate Euclidean triangle ABC,

$$a < B_r + C_r, \qquad b < C_r + A_r, \qquad c < A_r + B_r.$$

It is sufficient to prove $a < B_r + C_r$; the other two inequalities follow by cyclic permutation. From the definitions,

$$B_r^2 = (c-a)^2 + 4r^2, \qquad C_r^2 = (a-b)^2 + 4r^2.$$

The following exact identity holds: $4B_r^2C_r^2 - (a^2 - B_r^2 - C_r^2)^2 = 16r^2(s-a)^2$.

To verify it, set: $x = s - a, \qquad y = s - b, \qquad z = s - c.$

Then: $a = y + z, \qquad b = z + x, \qquad c = x + y,$

and Heron's identity gives $r^2 = \frac{xyz}{x+y+z}$.

Direct substitution and simplification reduce the left-hand side of the exact identity to

$$16x^2r^2 = 16r^2(s-a)^2.$$

Since the original triangle is nondegenerate, $r > 0$ and $s - a > 0$. Consequently,

$$4B_r^2C_r^2 - (a^2 - B_r^2 - C_r^2)^2 > 0.$$

Set: $X = a^2 - B_r^2 - C_r^2$. If $X \leq 0$, then, because $B_r, C_r > 0$,

$$a^2 \leq B_r^2 + C_r^2 < B_r^2 + C_r^2 + 2B_rC_r = (B_r + C_r)^2,$$

so $a < B_r + C_r$. If $X > 0$, the preceding strict identity implies

$$4B_r^2C_r^2 > X^2,$$

hence: $2B_rC_r > X$. Therefore: $(B_r + C_r)^2 = B_r^2 + C_r^2 + 2B_rC_r$,

and therefore $(B_r + C_r)^2 > B_r^2 + C_r^2 + X = a^2$. Since all quantities are positive,

$$a < B_r + C_r.$$

Cyclic permutation proves the remaining two inequalities.

1.4. Strict bilateral refinement chains

The classical triangle condition in the a-direction is $|\mathrm{b}-\mathrm{c}| < a < b + c$.

The lower refined relation gives $|\mathrm{b}-\mathrm{c}| < \mathrm{A_r} < a$,

whereas the cross-derived upper inequality gives: $\mathrm{a} < \mathrm{B_r} + \mathrm{C_r}$.

Since $\mathrm{B_r} < b$ and $\mathrm{C_r} < c$, $\mathrm{B_r} + \mathrm{C_r} < b + c$.

Combining the four strict comparisons yields the complete refinement chain

$$\boxed{|\mathrm{b}-\mathrm{c}| < \mathrm{A_r} < a < \mathrm{B_r} + \mathrm{C_r} < b + c}.$$

Cyclically,

$$\boxed{|\mathrm{c}-\mathrm{a}| < \mathrm{B_r} < b < \mathrm{C_r} + \mathrm{A_r} < c + a},$$

$$\boxed{|\mathrm{a}-\mathrm{b}| < \mathrm{C_r} < c < \mathrm{A_r} + \mathrm{B_r} < a + b}.$$

Consequently,

$$\boxed{(\mathrm{A_r}, \mathrm{B_r} + \mathrm{C_r}) \subsetneq (|\mathrm{b}-\mathrm{c}|, \mathrm{b}+\mathrm{c})}.$$

All quantities in each chain are evaluated from the same fixed triangle. The result is therefore a strict pointwise refinement of the classical side interval.

Denote by Δ_r the area of the triangle with side lengths $\mathrm{A_r}, \mathrm{B_r}, \mathrm{C_r}$, and by $\tilde{\Delta}_\mathrm{r}$ the area of the triangle with side lengths

$$\frac{\mathrm{A_r}}{2\mathrm{r}}, \qquad \frac{\mathrm{B_r}}{2\mathrm{r}}, \qquad \frac{\mathrm{C_r}}{2\mathrm{r}}.$$

2. Area of the Triangle with Side Lengths $\mathbf{A_r}, \mathbf{B_r}, \mathbf{C_r}$

For a triangle with side lengths $\mathrm{x}, \mathrm{y}, \mathrm{z}$, Heron's formula in algebraic form is

$$16\mathcal{A}^2 = 2\mathrm{x}^2\mathrm{y}^2 + 2\mathrm{y}^2\mathrm{z}^2 + 2\mathrm{z}^2\mathrm{x}^2 - \mathrm{x}^4 - \mathrm{y}^4 - \mathrm{z}^4. \qquad (2)$$

Apply (2) with $\mathrm{x} = \mathrm{A_r}$, $\mathrm{y} = \mathrm{B_r}$, and $\mathrm{z} = \mathrm{C_r}$. Introduce

$$\mathrm{p} = (\mathrm{b}-\mathrm{c})^2, \qquad \mathrm{q} = (\mathrm{c}-\mathrm{a})^2, \qquad \mathrm{t} = (\mathrm{a}-\mathrm{b})^2, \qquad \mathrm{h} = 4\mathrm{r}^2.$$

Then $\mathrm{A_r^2} = \mathrm{p} + \mathrm{h}$, $\mathrm{B_r^2} = \mathrm{q} + \mathrm{h}$, and $\mathrm{C_r^2} = \mathrm{t} + \mathrm{h}$, so

$$16\Delta_\mathrm{r}^2 = 2(\mathrm{p}+\mathrm{h})(\mathrm{q}+\mathrm{h}) + 2(\mathrm{q}+\mathrm{h})(\mathrm{t}+\mathrm{h}) + 2(\mathrm{t}+\mathrm{h})(\mathrm{p}+\mathrm{h})$$
$$-(\mathrm{p}+\mathrm{h})^2 - (\mathrm{q}+\mathrm{h})^2 - (\mathrm{t}+\mathrm{h})^2. \qquad (3)$$

Since $(b-c)+(c-a)+(a-b)=0$,

the identity $2(pq+qt+tp)-(p^2+q^2+t^2)=0$

follows. Therefore, (3) reduces to

$$16\Delta_r^2=2h(p+q+t)+3h^2,$$

$$16\Delta_r^2=8r^2[(a-b)^2+(b-c)^2+(c-a)^2]+48r^4,$$

$$16\Delta_r^2=16r^2\left[\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}+3r^2\right]. \qquad (4)$$

Since $\Delta_r>0$, we obtain

$$\Delta_r=r\sqrt{\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}+3r^2}. \qquad (5)$$

Using $\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}=a^2+b^2+c^2-ab-bc-ca$,

formula (5) becomes $\Delta_r=r\sqrt{a^2+b^2+c^2-ab-bc-ca+3r^2}$. (6)

Forms in Terms of Classical Invariants

The standard identities

$$a^2+b^2+c^2=2(s^2-r^2-4Rr),$$

$$ab+bc+ca=s^2+r^2+4Rr$$

give: $a^2+b^2+c^2-ab-bc-ca+3r^2=s^2-12Rr$.

Thus,

$$\Delta_r=r\sqrt{s^2-12Rr}. \qquad (7)$$

Since $\Delta=rs$, it also follows that

$$\Delta_r=\sqrt{\Delta^2-12Rr^3}. \qquad (8)$$

3. Area of the Normalized Triangle

Passing from the side lengths A_r, B_r, C_r to $\frac{A_r}{2r}$, $\frac{B_r}{2r}$, $\frac{C_r}{2r}$

is a similarity transformation with scale factor $\lambda=\frac{1}{2r}$.

The area is multiplied by λ^2, hence $\tilde{\Delta}_r=\frac{\Delta_r}{4r^2}$. (9)

From (5)–(8), we obtain the equivalent forms

$$\tilde{\Delta}_r = \frac{1}{4}\sqrt{3 + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2r^2}}. \qquad (10)$$

$$\tilde{\Delta}_r = \frac{1}{4r}\sqrt{a^2 + b^2 + c^2 - ab - bc - ca + 3r^2}. \qquad (11)$$

$$\tilde{\Delta}_r = \frac{1}{4}\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}}. \qquad (12)$$

$$\tilde{\Delta}_r = \frac{1}{4r^2}\sqrt{\Delta^2 - 12Rr^3}. \qquad (13)$$

The area $\tilde{\Delta}_r$ is dimensionless because the side lengths of the normalized triangle are dimensionless.

4. Summary

$$\Delta_r = r\sqrt{\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} + 3r^2} = r\sqrt{s^2 - 12Rr} = \sqrt{\Delta^2 - 12Rr^3},$$

$$\tilde{\Delta}_r = \frac{1}{4}\sqrt{3 + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2r^2}} = \frac{1}{4}\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}} = \frac{1}{4r^2}\sqrt{\Delta^2 - 12Rr^3}.$$

It is also proved in [1] that: $\frac{n_a}{h_a} = \frac{A_r}{2r}$ (and analogous);

$\frac{n_a}{h_a}, \frac{n_b}{h_b}, \frac{n_c}{h_c}$ can be sides of a triangle [1] and its area is: $\tilde{\Delta}_r = \frac{1}{4}\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}}$.(14)

5. Data and Notation

Let ABC be a nondegenerate Euclidean triangle with side lengths a, b, c, corresponding altitudes h_a, h_b, h_c, area $\Delta = [ABC]$, semiperimeter s, inradius r, and circumradius R.

We denote by n_a, n_b, n_c the lengths of the Nagel cevians corresponding to the vertices A, B, C. The classical relations between the area, side lengths, and altitudes are

$$2\Delta = ah_a = bh_b = ch_c.$$

6. Heron's Formula for the Side Lengths $\frac{n_a}{h_a}, \frac{n_b}{h_b}, \frac{n_c}{h_c}$

Consider the triangle whose side lengths are

$$x = \frac{n_a}{h_a}, \qquad y = \frac{n_b}{h_b}, \qquad z = \frac{n_c}{h_c}.$$

Its semiperimeter is

$$p_N = \frac{1}{2}\left(\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c}\right).$$

If Δ_N denotes the area of this triangle, Heron's formula gives

$$\Delta_N = \sqrt{p_N\left(p_N - \frac{n_a}{h_a}\right)\left(p_N - \frac{n_b}{h_b}\right)\left(p_N - \frac{n_c}{h_c}\right)}.$$

The equivalent form, without introducing the semiperimeter, is

$$\Delta_N = \frac{1}{4}\sqrt{\left(\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c}\right)\left(-\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c}\right)\left(\frac{n_a}{h_a} - \frac{n_b}{h_b} + \frac{n_c}{h_c}\right)\left(\frac{n_a}{h_a} + \frac{n_b}{h_b} - \frac{n_c}{h_c}\right)}.$$

$$\Delta_N = \frac{1}{4}\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}}. \ (15)$$

From (14) and (15): $\Delta_N = \frac{1}{4}\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}} = \tilde{\Delta}_r$

7. Bringing the Three Side Lengths to a Common Denominator

From $2\Delta = ah_a = bh_b = ch_c$ it follows that

$$h_a = \frac{2\Delta}{a}, \qquad h_b = \frac{2\Delta}{b}, \qquad h_c = \frac{2\Delta}{c}.$$

Therefore: $\frac{n_a}{h_a} = \frac{an_a}{2\Delta}, \qquad \frac{n_b}{h_b} = \frac{bn_b}{2\Delta}, \qquad \frac{n_c}{h_c} = \frac{cn_c}{2\Delta}.$

Thus: $\left(\frac{n_a}{h_a}, \frac{n_b}{h_b}, \frac{n_c}{h_c}\right) = \frac{1}{2\Delta}(an_a, bn_b, cn_c).$

The semiperimeter of the triangle with side lengths $\frac{n_a}{h_a}, \frac{n_b}{h_b}, \frac{n_c}{h_c}$ therefore becomes

$$p_N = \frac{an_a + bn_b + cn_c}{4\Delta}.$$

Substitution into Heron's formula yields

$$\Delta_N = \frac{1}{16\Delta^2}\sqrt{(an_a + bn_b + cn_c)(-an_a + bn_b + cn_c)(an_a - bn_b + cn_c)(an_a + bn_b - cn_c)}.$$

8. The Triangle with Side Lengths an_a, bn_b, cn_c

The triangle with side lengths: an_a, bn_b, cn_c

is obtained from the triangle with side lengths

$$\frac{n_a}{h_a}, \quad \frac{n_b}{h_b}, \quad \frac{n_c}{h_c}$$

by multiplying all side lengths by the positive factor 2Δ. Consequently, if the first triple forms a nondegenerate triangle, then an_a, bn_b, cn_c also form a nondegenerate triangle.

Let $\mathcal{A}_N$ denote the area of this new triangle. The areas of similar figures scale as the square of the similarity factor; therefore: $\mathcal{A}_N = (2\Delta)^2\Delta_N = 4\Delta^2\Delta_N$.

Using the simplified form of Δ_N, we obtain: $\mathcal{A}_N = 4\Delta^2 \cdot \frac{1}{4}\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}} = \Delta^2\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}}$. (16)

Applying Heron's formula directly to the side lengths an_a, bn_b, cn_c gives the equivalent form

$$\mathcal{A}_N = \frac{1}{4}\sqrt{(an_a + bn_b + cn_c)(-an_a + bn_b + cn_c)(an_a - bn_b + cn_c)(an_a + bn_b - cn_c)}.$$

Oppenheim's Weighted Triangle Inequality

Let x, y, and z be the side lengths of a nondegenerate Euclidean triangle. Let $\Delta[x, y, z] > 0$ denote its area, and let $\alpha, \beta, \gamma > 0$. Define

$$P = \alpha\beta + \beta\gamma + \gamma\alpha > 0.$$

Oppenheim's weighted triangle inequality states that

$$\alpha x^2 + \beta y^2 + \gamma z^2 \geq 4\Delta[x, y, z]\sqrt{\alpha\beta + \beta\gamma + \gamma\alpha}.[2]$$

Equality holds if and only if: $\frac{x^2}{\beta+\gamma} = \frac{y^2}{\gamma+\alpha} = \frac{z^2}{\alpha+\beta}$.

Oppenheim's Inequality for the Nagel-Cevian Ratios

Proposition 1. Let: $\frac{n_a}{h_a}$, $\frac{n_b}{h_b}$, $\frac{n_c}{h_c}$

be the side lengths of a nondegenerate triangle, and let $\mathit{\Delta}_N$ denote its area. For positive weights $\alpha,\beta,\gamma>0$, Oppenheim's weighted triangle inequality gives

$$\alpha\left(\frac{n_a}{h_a}\right)^2+\beta\left(\frac{n_b}{h_b}\right)^2+\gamma\left(\frac{n_c}{h_c}\right)^2\geq 4\mathit{\Delta}_N\sqrt{\alpha\beta+\beta\gamma+\gamma\alpha}. \quad (17)$$

Equality holds if and only if

$$\frac{\left(\frac{n_a}{h_a}\right)^2}{\beta+\gamma}=\frac{\left(\frac{n_b}{h_b}\right)^2}{\gamma+\alpha}=\frac{\left(\frac{n_c}{h_c}\right)^2}{\alpha+\beta}.$$

$\alpha,\beta,\gamma>0$, Oppenheim's weighted triangle inequality gives

$$\alpha\left(\frac{n_a}{h_a}\right)^2+\beta\left(\frac{n_b}{h_b}\right)^2+\gamma\left(\frac{n_c}{h_c}\right)^2\geq\sqrt{\frac{\mathrm{s}^2}{\mathrm{r}^2}-12\frac{\mathrm{R}}{\mathrm{r}}}\sqrt{\alpha\beta+\beta\gamma+\gamma\alpha}. \quad (18)$$

Unweighted case. For $\alpha=\beta=\gamma=1$, the inequality becomes

$$\left(\frac{n_a}{h_a}\right)^2+\left(\frac{n_b}{h_b}\right)^2+\left(\frac{n_c}{h_c}\right)^2\geq\sqrt{3}\sqrt{\frac{\mathrm{s}^2}{\mathrm{r}^2}-12\frac{\mathrm{R}}{\mathrm{r}}}. \quad (19)$$

In this case, equality holds if and only if $\frac{n_a}{h_a}=\frac{n_b}{h_b}=\frac{n_c}{h_c}$.

Proposition 2. Let $\mathrm{an_a}$, $\mathrm{bn_b}$, $\mathrm{cn_c}$

be the side lengths of a nondegenerate triangle, and let $\mathcal{A}_{\mathrm{N}}$ denote its area. For positive weights $\alpha,\beta,\gamma>0$, Oppenheim's weighted triangle inequality gives

$$\alpha(\mathrm{an_a})^2+\beta(\mathrm{bn_b})^2+\gamma(\mathrm{cn_c})^2\geq 4\mathcal{A}_{\mathrm{N}}\sqrt{\alpha\beta+\beta\gamma+\gamma\alpha}. \quad (20)$$

Equality holds if and only if

$$\frac{(\mathrm{an_a})^2}{\beta+\gamma}=\frac{(\mathrm{bn_b})^2}{\gamma+\alpha}=\frac{(\mathrm{cn_c})^2}{\alpha+\beta}$$

$\alpha,\beta,\gamma>0$, Oppenheim's weighted triangle inequality gives

$$\alpha(\mathrm{an_a})^2+\beta(\mathrm{bn_b})^2+\gamma(\mathrm{cn_c})^2\geq 4\Delta^2\sqrt{\frac{\mathrm{s}^2}{\mathrm{r}^2}-12\frac{\mathrm{R}}{\mathrm{r}}}\sqrt{\alpha\beta+\beta\gamma+\gamma\alpha} \quad (21)$$

References

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Complete Algebraic Calculation of Two Associated Triangle Areas (Appendix)

Let ABC be a nondegenerate Euclidean triangle with side lengths a, b, c, semiperimeter s, area Δ, inradius r, and circumradius R. Define

$$A_r = \sqrt{(b-c)^2 + 4r^2}, \qquad B_r = \sqrt{(c-a)^2 + 4r^2}, \qquad C_r = \sqrt{(a-b)^2 + 4r^2}.$$

Let Δ_r denote the area of the nondegenerate triangle with side lengths A_r, B_r, C_r.

1. Heron's Formula Written Directly with A_r, B_r, C_r

Heron's formula in algebraic form gives

$$16\Delta_r^2 = 2A_r^2B_r^2 + 2B_r^2C_r^2 + 2C_r^2A_r^2 - A_r^4 - B_r^4 - C_r^4. \qquad (A.1)$$

Introduce

$$u = b - c, \qquad v = c - a, \qquad w = a - b,$$

and

$$h = 4r^2.$$

Then

$$u + v + w = (b - c) + (c - a) + (a - b) = 0. \qquad (A.2)$$

Also,

$$A_r^2 = u^2 + h, \qquad B_r^2 = v^2 + h, \qquad C_r^2 = w^2 + h. \qquad (A.3)$$

2. Direct Substitution and Expansion

Substituting (A.3) into (A.1),

$$16\Delta_r^2 = 2(u^2 + h)(v^2 + h) + 2(v^2 + h)(w^2 + h)$$
$$+2(w^2 + h)(u^2 + h) - (u^2 + h)^2 - (v^2 + h)^2 - (w^2 + h)^2. \qquad (A.4)$$

The three positive products expand as follows:

$$2(u^2 + h)(v^2 + h) = 2u^2v^2 + 2hu^2 + 2hv^2 + 2h^2,$$
$$2(v^2 + h)(w^2 + h) = 2v^2w^2 + 2hv^2 + 2hw^2 + 2h^2,$$
$$2(w^2 + h)(u^2 + h) = 2w^2u^2 + 2hw^2 + 2hu^2 + 2h^2.$$

Therefore, their sum is

$$2(u^2v^2+v^2w^2+w^2u^2)+4h(u^2+v^2+w^2)+6h^2. \qquad (A.5)$$

The three square terms expand as

$$(u^2+h)^2=u^4+2hu^2+h^2,$$

$$(v^2+h)^2=v^4+2hv^2+h^2,$$

$$(w^2+h)^2=w^4+2hw^2+h^2.$$

Hence, their sum is

$$u^4+v^4+w^4+2h(u^2+v^2+w^2)+3h^2. \qquad (A.6)$$

Using (A.5) and (A.6) in (A.4), we obtain

$$16\Delta_r^2=2(u^2v^2+v^2w^2+w^2u^2)-(u^4+v^4+w^4)$$

$$+2h(u^2+v^2+w^2)+3h^2. \qquad (A.7)$$

3. Cancellation of the Fourth-Degree Terms

Because $u+v+w=0$, squaring this equality gives

$$u^2+v^2+w^2+2uv+2vw+2wu=0,$$

hence

$$uv+vw+wu=-\frac{u^2+v^2+w^2}{2}. \qquad (A.8)$$

Now expand the square of $uv+vw+wu$:

$$(uv+vw+wu)^2=u^2v^2+v^2w^2+w^2u^2+2uvw(u+v+w).$$

Since $u+v+w=0$,

$$(uv+vw+wu)^2=u^2v^2+v^2w^2+w^2u^2. \qquad (A.9)$$

Using (A.8),

$$u^2v^2+v^2w^2+w^2u^2=\frac{(u^2+v^2+w^2)^2}{4}. \qquad (A.10)$$

On the other hand,

$$(u^2+v^2+w^2)^2=u^4+v^4+w^4+2(u^2v^2+v^2w^2+w^2u^2).$$

By (A.10), the left-hand side equals

$$4(u^2v^2+v^2w^2+w^2u^2).$$

Therefore,

$$u^4 + v^4 + w^4 = 2(u^2v^2 + v^2w^2 + w^2u^2),$$

so

$$2(u^2v^2 + v^2w^2 + w^2u^2) - (u^4 + v^4 + w^4) = 0. \qquad (A.11)$$

4. Area of the Triangle with Sides $\mathbf{A_r}, \mathbf{B_r}, \mathbf{C_r}$

Using (A.11), formula (A.7) reduces to

$$16\Delta_r^2 = 2h(u^2 + v^2 + w^2) + 3h^2. \qquad (A.12)$$

Since $h = 4r^2$,

$$16\Delta_r^2 = 2(4r^2)(u^2 + v^2 + w^2) + 3(4r^2)^2,$$

$$16\Delta_r^2 = 8r^2(u^2 + v^2 + w^2) + 48r^4,$$

$$16\Delta_r^2 = 16r^2\left(\frac{u^2 + v^2 + w^2}{2} + 3r^2\right).$$

Dividing by 16,

$$\Delta_r^2 = r^2\left(\frac{u^2 + v^2 + w^2}{2} + 3r^2\right). \qquad (A.13)$$

Because $r > 0$ and $\Delta_r > 0$, the positive square root gives

$$\Delta_r = r\sqrt{\frac{u^2 + v^2 + w^2}{2} + 3r^2}. \qquad (A.14)$$

Returning to the original side lengths: $u^2 + v^2 + w^2 = (b - c)^2 + (c - a)^2 + (a - b)^2$.

Thus: $\Delta_r = r\sqrt{\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} + 3r^2}. \qquad (A.15)$

5. Equivalent Symmetric Form

Expand the sum of squared differences:

$(a - b)^2 + (b - c)^2 + (c - a)^2$=$(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)$.

and therefore

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 2(a^2 + b^2 + c^2 - ab - bc - ca).$$

Hence,

$$\Delta_r = r\sqrt{a^2 + b^2 + c^2 - ab - bc - ca + 3r^2}. \qquad (A.16)$$

6. Form in Terms of s, r, R

Use the classical identities: $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$,

and $ab + bc + ca = s^2 + r^2 + 4Rr$.

Then the expression $a^2 + b^2 + c^2 - ab - bc - ca + 3r^2$

becomes $2(s^2 - r^2 - 4Rr) - (s^2 + r^2 + 4Rr) + 3r^2$.

Expanding the parentheses gives $2s^2 - 2r^2 - 8Rr - s^2 - r^2 - 4Rr + 3r^2$.

Collecting like terms gives $s^2 - 12Rr$. Hence: $\Delta_r = r\sqrt{s^2 - 12Rr}$. $\qquad (A.17)$

Because $\Delta = rs$, $\Delta^2 = r^2 s^2$. Squaring (A.17),

$$\Delta_r^2 = r^2(s^2 - 12Rr), \Delta_r^2 = r^2 s^2 - 12Rr^3, \Delta_r^2 = \Delta^2 - 12Rr^3.$$

Therefore: $\Delta_r = \sqrt{\Delta^2 - 12Rr^3}$. $\qquad (A.18)$

7. Area of the Normalized Triangle

Consider the triangle with side lengths

$$\frac{A_r}{2r}, \quad \frac{B_r}{2r}, \quad \frac{C_r}{2r}.$$

The similarity scale factor is $\lambda = \frac{1}{2r}$. Areas scale by λ^2, so $\quad \tilde{\Delta}_r = \frac{\Delta_r}{4r^2}$. $\qquad (A.19)$

7.1. First Normalized Form

Using (A.15), $\tilde{\Delta}_r = \frac{1}{4r^2}\, r\sqrt{\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} + 3r^2}$,

and therefore: $\tilde{\Delta}_r = \frac{1}{4r}\sqrt{\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} + 3r^2}$.

Factor r^2 inside the square root. The expression $\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} + 3r^2$

is equal to $r^2\left[3 + \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2r^2}\right]$. Since $r > 0$, $\sqrt{r^2} = r$. Hence,

$$\tilde{\Delta}_r = \frac{1}{4}\sqrt{3 + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2r^2}}. \qquad (A.20)$$

7.2. Symmetric Normalized Form

Using (A.16),

$$\tilde{\Delta}_r = \frac{1}{4r}\sqrt{a^2 + b^2 + c^2 - ab - bc - ca + 3r^2}. \qquad (A.21)$$

7.3. Normalized Form in Terms of s, r, R

Using (A.17),

$$\tilde{\Delta}_r = \frac{1}{4r}\sqrt{s^2 - 12Rr}.$$

Because $r > 0$, this is equivalently $\tilde{\Delta}_r = \frac{1}{4}\sqrt{\frac{s^2-12Rr}{r^2}}$, and therefore $\tilde{\Delta}_r = \frac{1}{4}\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}}$. $\qquad$ (A.22)

7.4. Normalized Form in Terms of the Original Area

Using (A.18),

$$\tilde{\Delta}_r = \frac{1}{4r^2}\sqrt{\Delta^2 - 12Rr^3}. \qquad (A.23)$$

8. Final Results

$$\Delta_r = r\sqrt{\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} + 3r^2},$$

$$\Delta_r = r\sqrt{a^2 + b^2 + c^2 - ab - bc - ca + 3r^2},$$

$$\Delta_r = r\sqrt{s^2 - 12Rr}, \Delta_r = \sqrt{\Delta^2 - 12Rr^3}.$$

For the normalized triangle,

$$\tilde{\Delta}_r = \frac{1}{4}\sqrt{3 + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2r^2}},$$

$$\tilde{\Delta}_r = \frac{1}{4r}\sqrt{a^2 + b^2 + c^2 - ab - bc - ca + 3r^2},$$

$$\tilde{\Delta}_r = \frac{1}{4}\sqrt{\frac{s^2}{r^2} - 12\frac{R}{r}}, \qquad \tilde{\Delta}_r = \frac{1}{4r^2}\sqrt{\Delta^2 - 12Rr^3}.$$