

Proposed Problem

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Problem

Let $a, b > 0$ and $\lambda \geq 2$. Prove that

$$\frac{\lambda a + b}{\sqrt[3]{a^2 b}} + \frac{a + \lambda b}{\sqrt[3]{a b^2}} \geq 4 + \sqrt{\frac{\lambda a^2 + b^2}{a(a + \lambda b)}} + \sqrt{\frac{a^2 + \lambda b^2}{b(b + \lambda a)}}.$$

Solution

We first prove the following auxiliary inequality: for every $x > 0$ and $\lambda \geq 2$,

$$\lambda x + \frac{1}{x^2} \geq 2 + \sqrt{\frac{\lambda x^6 + 1}{x^3(x^3 + \lambda)}}. \quad (1)$$

Set

$$R_\lambda(x) = \sqrt{\frac{\lambda x^6 + 1}{x^3(x^3 + \lambda)}} \quad \text{and} \quad F_\lambda(x) = \lambda x + \frac{1}{x^2} - 2 - R_\lambda(x).$$

Differentiation with respect to λ gives

$$\frac{\partial R_\lambda(x)}{\partial \lambda} = \frac{x^9 - 1}{2R_\lambda(x)x^3(x^3 + \lambda)^2}.$$

If $0 < x \leq 1$, this derivative is nonpositive, and hence

$$\frac{\partial F_\lambda(x)}{\partial \lambda} = x - \frac{\partial R_\lambda(x)}{\partial \lambda} > 0.$$

If $x \geq 1$, then

$$R_\lambda(x)^2 = \frac{\lambda x^6 + 1}{x^3(x^3 + \lambda)} \geq \frac{\lambda x^3}{x^3 + \lambda}.$$

Consequently,

$$\begin{aligned} 0 \leq \frac{\partial R_\lambda(x)}{\partial \lambda} &\leq \frac{x^9}{2\sqrt{\lambda x^3/(x^3 + \lambda)} x^3(x^3 + \lambda)^2} \\ &= \frac{x^{9/2}}{2\sqrt{\lambda} (x^3 + \lambda)^{3/2}} \leq \frac{1}{2\sqrt{\lambda}} < x. \end{aligned}$$

Thus $\partial F_\lambda(x)/\partial \lambda > 0$ also when $x \geq 1$. It is therefore enough to verify (1) for $\lambda = 2$.

The expression $2x + x^{-2} - 2$ is positive (in fact, its minimum is 1, attained at $x = 1$). Direct expansion yields

$$\begin{aligned} & \left(2x + \frac{1}{x^2} - 2\right)^2 - \frac{2x^6 + 1}{x^3(x^3 + 2)} \\ &= \frac{(x-1)^2 P(x)}{x^4(x^3 + 2)}, \end{aligned}$$

where

$$P(x) = 4x^7 - 2x^5 + 8x^4 - 2x^3 - 4x^2 + 3x + 2.$$

We check that $P(x) > 0$. If $x \geq 1$, then

$$P(x) = 2x^5(2x^2 - 1) + 2x^2(4x^2 - x - 2) + 3x + 2 > 0.$$

If $0 < x \leq 1$, then $x^5 \leq x^2$ and $x^3 \leq x^2$, so

$$\begin{aligned} P(x) &\geq 8x^4 - 8x^2 + 3x + 2 \\ &= 2 + 3x - 8x^2(1 - x^2) \\ &\geq 3x > 0, \end{aligned}$$

because $x^2(1 - x^2) \leq 1/4$. Hence the difference of the two squares is nonnegative, proving (1).

Now put

$$x = \sqrt[3]{\frac{a}{b}}.$$

Then

$$\frac{\lambda a + b}{\sqrt[3]{a^2 b}} = \lambda x + \frac{1}{x^2}$$

and

$$\sqrt{\frac{\lambda a^2 + b^2}{a(a + \lambda b)}} = \sqrt{\frac{\lambda x^6 + 1}{x^3(x^3 + \lambda)}}.$$

Applying (1) therefore gives

$$\frac{\lambda a + b}{\sqrt[3]{a^2 b}} \geq 2 + \sqrt{\frac{\lambda a^2 + b^2}{a(a + \lambda b)}}. \quad (2)$$

Interchanging a and b in (2), we likewise obtain

$$\frac{a + \lambda b}{\sqrt[3]{a b^2}} \geq 2 + \sqrt{\frac{a^2 + \lambda b^2}{b(b + \lambda a)}}. \quad (3)$$

Adding (2) and (3) proves the required inequality.

The equality conditions in the auxiliary inequality show that equality is possible only when $\lambda = 2$ and $x = 1$. Thus equality holds precisely for

$$\boxed{\lambda = 2 \quad \text{and} \quad a = b}.$$