

Proposed Problem

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Problem

Let $A_0A_1 \dots A_{n-1}$ be a regular n -gon of circumradius R , where $n \geq 4$. For

$$0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

let $a_k = A_0A_k$; in particular, $a_0 = 0$. Prove the following general relations among its chord lengths.

1. For every $1 \leq k < \lfloor n/2 \rfloor$,

$$a_{k+1} = \frac{a_2}{a_1} a_k - a_{k-1}.$$

2. If $1 \leq v \leq u$ and $u + v \leq \lfloor n/2 \rfloor$, then

$$a_{u+v}a_{u-v} = a_u^2 - a_v^2.$$

3. Let m be a nonnegative integer and let $1 \leq r \leq \lfloor n/2 \rfloor$. Put

$$h_j = 2m + 1 - 2j \quad (j = 0, 1, \dots, m).$$

Prove the odd-power sum formula

$$\sum_{k=1}^r a_k^{2m+1} = \frac{(2R)^{2m+1}}{2^{2m}} \sum_{j=0}^m (-1)^{m+j} \binom{2m+1}{j} \times \frac{\sin \frac{h_j r \pi}{2n} \sin \frac{h_j (r+1) \pi}{2n}}{\sin \frac{h_j \pi}{2n}}.$$

In particular, derive explicit formulas for the sum of the first r chord lengths and for the sum of their cubes.

Solution

Set

$$\theta = \frac{\pi}{n}.$$

The chord A_0A_k subtends the central angle $2k\theta$. Bisecting the corresponding isosceles triangle gives the fundamental formula

$$a_k = 2R \sin(k\theta). \quad (1)$$

1. Recurrence relation

The sine addition formulas imply

$$\sin((k+1)\theta) + \sin((k-1)\theta) = 2 \cos \theta \sin(k\theta).$$

Multiplying by $2R$ and using (1), we obtain

$$a_{k+1} + a_{k-1} = 2 \cos \theta a_k. \quad (2)$$

Moreover,

$$\frac{a_2}{a_1} = \frac{\sin 2\theta}{\sin \theta} = 2 \cos \theta.$$

Substitution into (2) yields

$$\boxed{a_{k+1} = \frac{a_2}{a_1} a_k - a_{k-1}}.$$

Thus all the chord lengths are generated from a_0, a_1, a_2 by one second-order recurrence.

2. Product relation

For arbitrary real numbers x and y , the product-to-sum formulas give

$$\begin{aligned} \sin(x+y) \sin(x-y) &= \frac{\cos 2y - \cos 2x}{2} \\ &= \sin^2 x - \sin^2 y. \end{aligned}$$

Take $x = u\theta$ and $y = v\theta$. Multiplication by $(2R)^2$ and use of (1) give

$$\boxed{a_{u+v} a_{u-v} = a_u^2 - a_v^2}.$$

For example, setting $v = 1$ gives the relation

$$a_{k+1} a_{k-1} = a_k^2 - a_1^2,$$

valid for every admissible k . Hence the identity relates arbitrarily distant chord lengths, not merely four fixed chords.

3. Sums of arbitrary odd powers

We first establish the standard expansion

$$\sin^{2m+1} x = \frac{1}{2^{2m}} \sum_{j=0}^m (-1)^{m+j} \binom{2m+1}{j} \sin((2m+1-2j)x). \quad (3)$$

Indeed, from

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

and the binomial theorem,

$$\sin^{2m+1} x = \frac{1}{(2i)^{2m+1}} \sum_{j=0}^{2m+1} (-1)^j \binom{2m+1}{j} e^{i(2m+1-2j)x}.$$

Pairing the term indexed by j with the term indexed by $2m+1-j$ converts each pair of exponentials into a sine and gives (3).

We also need the finite sine-sum identity

$$\sum_{k=1}^r \sin(kt) = \frac{\sin \frac{rt}{2} \sin \frac{(r+1)t}{2}}{\sin \frac{t}{2}}. \quad (4)$$

To verify it, sum the telescoping identities

$$2 \sin(kt) \sin \frac{t}{2} = \cos \left(k - \frac{1}{2} \right) t - \cos \left(k + \frac{1}{2} \right) t$$

from $k = 1$ to $k = r$, and then use the difference-of-cosines formula.

By (1),

$$\sum_{k=1}^r a_k^{2m+1} = (2R)^{2m+1} \sum_{k=1}^r \sin^{2m+1}(k\theta). \quad (5)$$

Apply (3) to every term on the right. With $h_j = 2m+1-2j$, interchange the two finite sums to obtain

$$\sum_{k=1}^r \sin^{2m+1}(k\theta) = \frac{1}{2^{2m}} \sum_{j=0}^m (-1)^{m+j} \binom{2m+1}{j} \sum_{k=1}^r \sin(kh_j\theta).$$

Formula (4), used with $t = h_j\theta$, now gives

$$\sum_{k=1}^r \sin(kh_j\theta) = \frac{\sin \frac{h_j r \theta}{2} \sin \frac{h_j (r+1) \theta}{2}}{\sin \frac{h_j \theta}{2}}.$$

The denominator cannot vanish: h_j is positive and odd, whereas a zero would require h_j to be a multiple of the even number $2n$. Substituting this expression into (5) and using $\theta = \pi/n$ proves the boxed odd-power formula.

For $m = 0$, we have $h_0 = 1$, and the general result reduces to

$$\sum_{k=1}^r a_k = 2R \frac{\sin \frac{r\pi}{2n} \sin \frac{(r+1)\pi}{2n}}{\sin \frac{\pi}{2n}}.$$

For $m = 1$, the two frequencies are $h_0 = 3$ and $h_1 = 1$. Therefore

$$\sum_{k=1}^r a_k^3 = 2R^3 \left(3 \frac{\sin \frac{r\pi}{2n} \sin \frac{(r+1)\pi}{2n}}{\sin \frac{\pi}{2n}} - \frac{\sin \frac{3r\pi}{2n} \sin \frac{3(r+1)\pi}{2n}}{\sin \frac{3\pi}{2n}} \right).$$

These are the requested explicit first-power and cubic relations.