

Proposed Problem

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Problem

Let A, B, p, q, μ be positive real numbers. Determine all real values of s for which the integral

$$M(s) = \int_{-\infty}^{\infty} \frac{e^{sx}}{(Ae^{px} + Be^{-qx})^\mu} dx$$

converges, and evaluate it in closed form.

Solution

We begin by determining the convergence range. As $x \rightarrow +\infty$,

$$\frac{e^{sx}}{(Ae^{px} + Be^{-qx})^\mu} \sim A^{-\mu} e^{(s-p\mu)x}.$$

Thus the integral converges at $+\infty$ precisely when $s < p\mu$. Similarly, as $x \rightarrow -\infty$,

$$\frac{e^{sx}}{(Ae^{px} + Be^{-qx})^\mu} \sim B^{-\mu} e^{(s+q\mu)x},$$

so convergence at $-\infty$ is equivalent to $s > -q\mu$. Consequently, the integral converges exactly for

$$\boxed{-q\mu < s < p\mu.}$$

Assume from now on that s belongs to this interval. Put

$$r = p + q, \quad u = \frac{s + q\mu}{r}.$$

The convergence condition is equivalent to $0 < u < \mu$. Factoring e^{-qx} from the expression in the denominator gives

$$Ae^{px} + Be^{-qx} = e^{-qx}(Ae^{rx} + B).$$

Therefore

$$M(s) = \int_{-\infty}^{\infty} \frac{e^{(s+q\mu)x}}{(Ae^{rx} + B)^\mu} dx.$$

We now make the substitution

$$t = \frac{A}{B}e^{rx}.$$

Then t ranges over $(0, \infty)$, and

$$dx = \frac{dt}{rt}, \quad e^{(s+q\mu)x} = \left(\frac{B}{A}t\right)^u, \quad Ae^{rx} + B = B(1+t).$$

Hence

$$\begin{aligned} M(s) &= \frac{1}{r} A^{-u} B^{u-\mu} \int_0^\infty t^{u-1} (1+t)^{-\mu} dt \\ &= \frac{1}{r} A^{-u} B^{u-\mu} B(u, \mu-u). \end{aligned}$$

Because $0 < u < \mu$, the beta integral is convergent. Using

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$$

we finally obtain

$$\boxed{M(s) = \frac{A^{-u} B^{u-\mu}}{p+q} \frac{\Gamma(u)\Gamma(\mu-u)}{\Gamma(\mu)}, \quad u = \frac{s+q\mu}{p+q}, \quad -q\mu < s < p\mu.}$$