

Proposed Problem

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Problem

Let ABC be a triangle. Prove that

$$\sum_{\text{cyc}} \frac{c(a-b)^2}{ab(a+b)} = \frac{s^4 + (2r^2 - 16Rr)s^2 + r^4 + 16Rr^3 + 12R^2r^2}{2Rr(s^2 + r^2 + 2Rr)}, \quad (1)$$

$$\sum_{\text{cyc}} \frac{r_c(r_a - r_b)^2}{r_a r_b (r_a + r_b)} = \frac{(4R^2 + 6Rr - r^2)s^2 - r(4R + r)^3}{Rrs^2}. \quad (2)$$

Here a, b, c are the side lengths of the triangle, s is its semiperimeter, R is its circumradius, r is its inradius, and r_a, r_b, r_c are its exradii. Both sums are cyclic.

Solution

Let Δ denote the area of the triangle, and introduce the positive variables

$$x = s - a, \quad y = s - b, \quad z = s - c.$$

Then

$$a = y + z, \quad b = z + x, \quad c = x + y, \quad x + y + z = s.$$

Furthermore, let

$$q = xy + yz + zx, \quad p = xyz.$$

By Heron's formula and the identity $\Delta = rs$, we have

$$p = (s - a)(s - b)(s - c) = \frac{\Delta^2}{s} = r^2 s. \quad (3)$$

On the other hand,

$$(x + y)(y + z)(z + x) = (x + y + z)(xy + yz + zx) - xyz = sq - p.$$

Since $(x + y)(y + z)(z + x) = abc$ and $abc = 4R\Delta = 4Rrs$, together with (3) this gives

$$sq - p = 4Rrs, \quad \text{hence} \quad q = r^2 + 4Rr. \quad (4)$$

We first establish the two algebraic identities needed below. With $s = x + y + z$, $q = xy + yz + zx$, and $p = xyz$, bringing the terms to a common denominator gives

$$\begin{aligned} \sum_{\text{cyc}} \frac{(x+y)(x-y)^2}{(y+z)(z+x)(s+z)} \\ = \frac{4s^6 - 16s^4q + 24s^3p + 3s^2q^2 + 10sqp - 9p^2}{2s^4q - 2s^3p + s^2q^2 - p^2}, \end{aligned} \quad (5)$$

$$\sum_{\text{cyc}} \frac{(x-y)^2}{z(x+y)} = \frac{s^2q^2 + 4sqp - 4q^3 - 9p^2}{p(sq - p)}. \quad (6)$$

Here are the details of the algebraic computation. A common denominator of the left-hand side of (5) is

$$\begin{aligned} D_1 &= (x+y)(y+z)(z+x)(s+x)(s+y)(s+z) \\ &= (sq - p)(2s^3 + sq + p) \\ &= 2s^4q - 2s^3p + s^2q^2 - p^2. \end{aligned}$$

The corresponding numerator is

$$N_1 = \sum_{\text{cyc}} (x+y)^2(x-y)^2(s+x)(s+y).$$

Expanding the three terms and collecting them in terms of $x + y + z = s$, $xy + yz + zx = q$, and $xyz = p$ yields

$$N_1 = 4s^6 - 16s^4q + 24s^3p + 3s^2q^2 + 10sqp - 9p^2,$$

which proves (5). Similarly, for the left-hand side of (6),

$$\begin{aligned} D_2 &= xyz(x+y)(y+z)(z+x) = p(sq - p), \\ N_2 &= \sum_{\text{cyc}} xy(x-y)^2(y+z)(z+x) \\ &= s^2q^2 + 4sqp - 4q^3 - 9p^2. \end{aligned}$$

This also proves (6).

The first identity

Expressing the side lengths in terms of x, y, z , we obtain

$$\sum_{\text{cyc}} \frac{c(a-b)^2}{ab(a+b)} = \sum_{\text{cyc}} \frac{(x+y)(x-y)^2}{(y+z)(z+x)(s+z)}.$$

Apply (5), and then substitute the values from (3) and (4). The numerator becomes

$$\begin{aligned} N_1 &= 4s^6 - 16s^4q + 24s^3p + 3s^2q^2 + 10sqp - 9p^2 \\ &= 4s^2(s^4 + (2r^2 - 16Rr)s^2 + r^4 + 16Rr^3 + 12R^2r^2), \end{aligned}$$

whereas the denominator becomes

$$\begin{aligned} D_1 &= 2s^4q - 2s^3p + s^2q^2 - p^2 \\ &= 8Rrs^2(s^2 + r^2 + 2Rr). \end{aligned}$$

Cancelling $4s^2$ from N_1/D_1 gives precisely the right-hand side of (1). This proves the first identity.

The second identity

The exradius opposite A satisfies $r_a = \Delta/(s - a)$, with analogous formulas for r_b and r_c . Since $\Delta = rs$, it follows that

$$r_a = \frac{rs}{x}, \quad r_b = \frac{rs}{y}, \quad r_c = \frac{rs}{z}.$$

Thus, in each cyclic term, all factors rs cancel:

$$\begin{aligned} \frac{r_c(r_a - r_b)^2}{r_a r_b(r_a + r_b)} &= \frac{\frac{rs}{z} \left(\frac{rs}{x} - \frac{rs}{y} \right)^2}{\frac{rs}{x} \frac{rs}{y} \left(\frac{rs}{x} + \frac{rs}{y} \right)} \\ &= \frac{(x - y)^2}{z(x + y)}. \end{aligned}$$

Consequently, by (6),

$$\sum_{\text{cyc}} \frac{r_c(r_a - r_b)^2}{r_a r_b(r_a + r_b)} = \frac{s^2 q^2 + 4sqp - 4q^3 - 9p^2}{p(sq - p)}.$$

After substituting (3)–(4), the numerator is

$$\begin{aligned} N_2 &= s^2 q^2 + 4sqp - 4q^3 - 9p^2 \\ &= 4r^2((4R^2 + 6Rr - r^2)s^2 - r(4R + r)^3), \end{aligned}$$

and the denominator is

$$D_2 = p(sq - p) = r^2 s \cdot 4Rrs = 4Rr^3 s^2.$$

Therefore,

$$\sum_{\text{cyc}} \frac{r_c(r_a - r_b)^2}{r_a r_b(r_a + r_b)} = \frac{(4R^2 + 6Rr - r^2)s^2 - r(4R + r)^3}{Rr s^2},$$

which is (2). This proves the second identity as well.