Proposed Problem

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Problem. Let $a, b, c > 0$ with $abc = 1$, and let p, q, s, r be real numbers satisfying $p > 0$, $q \geq s \geq 0$, and $r \geq p + q$. Set

$$M = 3p + q + s, \qquad T_r = a^r + b^r + c^r, \qquad \Delta_r = \frac{1}{2}\sum_{\text{cyc}}(a^r - b^r)^2.$$

Prove that $3rT_r^2 - M\Delta_r > 0$ and

$$\boxed{\sum_{\text{cyc}} \frac{a^r}{(b^p + c^p)(c^q + c^s + 1)} \geq \frac{1}{2} + \frac{M\Delta_r}{2(3rT_r^2 - M\Delta_r)} \geq \frac{1}{2}.} \tag{1}$$

Determine all equality cases and show that the constant $1/2$ is best possible. All cyclic sums are taken over (a, b, c), (b, c, a), and (c, a, b).

Solution. A monomial estimate under $abc = 1$

Lemma 1. *Let $a, b, c > 0$, $abc = 1$, and $r > 0$. If $u, v \geq 0$ and $u + v \leq r$, then*

$$b^u c^v \leq \frac{r - u - v}{3r}\, a^r + \frac{r + 2u - v}{3r}\, b^r + \frac{r - u + 2v}{3r}\, c^r. \tag{2}$$

Proof. Put $x = a^r$, $y = b^r$, and $z = c^r$, so that $xyz = 1$. Define

$$\lambda = \frac{r - u - v}{3r}, \qquad \mu = \lambda + \frac{u}{r}, \qquad \nu = \lambda + \frac{v}{r}.$$

These weights are nonnegative and satisfy

$$\lambda + \mu + \nu = \frac{r - u - v}{r} + \frac{u + v}{r} = 1.$$

Moreover,

$$x^\lambda y^\mu z^\nu = (xyz)^\lambda y^{u/r} z^{v/r} = b^u c^v.$$

The weighted arithmetic–geometric mean inequality gives

$$b^u c^v = x^\lambda y^\mu z^\nu \leq \lambda x + \mu y + \nu z,$$

which is (2). Terms with zero weight are simply omitted when applying the weighted inequality. $\square$

A common upper bound for the denominators

Keep $x = a^r$, $y = b^r$, and $z = c^r$, and denote the left-hand side of (1) by F. Expanding the first denominator yields

$$(b^p + c^p)(c^q + c^s + 1) = b^p c^q + c^{p+q} + b^p c^s + c^{p+s} + b^p + c^p. \tag{3}$$

The corresponding exponent pairs (u, v) are

$$(p, q), \quad (0, p+q), \quad (p, s), \quad (0, p+s), \quad (p, 0), \quad (0, p).$$

Each satisfies $u + v \le p + q \le r$, so Lemma 1 applies to all six monomials. Summing their exponents gives

$$\sum u = 3p, \qquad \sum v = 3p + 2q + 2s, \qquad \sum (u + v) = 6p + 2q + 2s = 2M.$$

Adding the six estimates therefore gives

$$(b^p + c^p)(c^q + c^s + 1) \le Ax + By + Cz, \tag{4}$$

where

$$A = 2 - \frac{2M}{3r}, \qquad B = A + \frac{3p}{r}, \qquad C = A + \frac{3p + 2q + 2s}{r}. \tag{5}$$

In particular,

$$A + B + C = 3A + \frac{2M}{r} = 6, \qquad B + C = 6 - A. \tag{6}$$

Since

$$0 < M = 3p + q + s \le 3p + 3q \le 3r,$$

we have $A \ge 0$ and $B, C > 0$. Thus all the new denominators are positive. The other two cyclic estimates are

$$(c^p + a^p)(a^q + a^s + 1) \le Ay + Bz + Cx,$$
$$(a^p + b^p)(b^q + b^s + 1) \le Az + Bx + Cy.$$

Replacing a positive denominator by an upper bound decreases the fraction. Consequently,

$$F \ge \frac{x}{Ax + By + Cz} + \frac{y}{Ay + Bz + Cx} + \frac{z}{Az + Bx + Cy}. \tag{7}$$

Cauchy–Schwarz and the remainder term

We use the Engel form of the Cauchy–Schwarz inequality:

$$\frac{\xi_1^2}{\eta_1} + \frac{\xi_2^2}{\eta_2} + \frac{\xi_3^2}{\eta_3} \ge \frac{(\xi_1 + \xi_2 + \xi_3)^2}{\eta_1 + \eta_2 + \eta_3} \qquad (\eta_1, \eta_2, \eta_3 > 0).$$

Writing the first fraction in (7) as $x^2/[x(Ax + By + Cz)]$, and similarly for the other two, we obtain

$$F \ge \frac{(x + y + z)^2}{x(Ax + By + Cz) + y(Ay + Bz + Cx) + z(Az + Bx + Cy)}. \tag{8}$$

Set

$$Q = x^2 + y^2 + z^2, \qquad P = xy + yz + zx.$$

The denominator in (8) is

$$AQ + (B + C)P = AQ + (6 - A)P.$$

Also,

$$T_r^2 = Q + 2P, \qquad \Delta_r = Q - P = \frac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2}.$$

Substituting the value of A gives

$$\begin{aligned} AQ + (6-A)P &= \left(2 - \frac{2M}{3r}\right) Q + \left(4 + \frac{2M}{3r}\right) P \\ &= 2(Q+2P) - \frac{2M}{3r}(Q-P) \\ &= \frac{2}{3r}(3rT_r^2 - M\Delta_r). \end{aligned}$$

The required denominator is strictly positive: since $P > 0$ and $M \le 3r$,

$$3rT_r^2 - M\Delta_r = (3r - M)Q + (6r + M)P > 0. \tag{9}$$

It now follows from (8) that

$$F \ge \frac{3rT_r^2}{2(3rT_r^2 - M\Delta_r)} = \frac{1}{2} + \frac{M\Delta_r}{2(3rT_r^2 - M\Delta_r)} \ge \frac{1}{2},$$

because $M > 0$ and $\Delta_r \ge 0$. This proves (1).

Equality cases and sharpness

If $F = 1/2$, the nonnegative remainder term must vanish. Hence $\Delta_r = 0$, which is equivalent to

$$x = y = z, \qquad \text{or, equivalently,} \qquad a^r = b^r = c^r.$$

As $r > 0$, this implies $a = b = c$. Together with $abc = 1$, it gives $a = b = c = 1$. Conversely, at $a = b = c = 1$, each fraction is $1/6$ and $\Delta_r = 0$, so equality holds throughout (1).

The first inequality in (1) also has no other equality cases. To see this, distinguish two possibilities.

- If $r > p$, the weighted AM–GM estimate for the term b^p in (3) has three positive weights:

  $$\lambda = \nu = \frac{r-p}{3r} > 0, \qquad \mu = \frac{r+2p}{3r} > 0.$$

  Equality requires $x = y = z$. Otherwise, (4) is strict, and so is the first inequality in (1).

- If $r = p$, the assumptions $r \ge p + q$ and $q \ge s \ge 0$ force $q = s = 0$. In this case $A = 0$ and $B = C = 3$, and the denominator estimates are exact. Equality in Cauchy–Schwarz holds precisely when

  $$\frac{1}{3(y+z)} = \frac{1}{3(z+x)} = \frac{1}{3(x+y)},$$

  which again is equivalent to $x = y = z$.

Thus equality in either inequality of (1) holds if and only if $a = b = c = 1$.

Finally, for every fixed admissible choice of p, q, s, r, the left-hand side equals $1/2$ at $a = b = c = 1$. Therefore the constant $1/2$ cannot be replaced by any larger constant and is best possible.