Proposed Problem

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Problem

Let ABC be any scalene triangle. Let $a = BC$, $b = CA$, and $c = AB$ denote its side lengths, and let A, B, and C denote the corresponding opposite interior angles. Prove that

$$\boxed{\sum_{\text{cyc}} \left(\frac{ab\cos C}{a-b}\right)^2 = (a+b+c)^2 + \left(\sum_{\text{cyc}} \frac{ab\cos C}{a-b}\right)^2.} \tag{1}$$

Solution

1. Expressing the terms in side lengths

By the law of cosines,

$$2ab\cos C = a^2 + b^2 - c^2, \qquad 2bc\cos A = b^2 + c^2 - a^2, \qquad 2ca\cos B = c^2 + a^2 - b^2.$$

Introduce the notation

$$x = \frac{ab\cos C}{a-b}, \qquad y = \frac{bc\cos A}{b-c}, \qquad z = \frac{ca\cos B}{c-a},$$

and set

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad T = a^2+b^2+c^2, \qquad D = (a-b)(b-c)(c-a).$$

Then $D \neq 0$, $T + 2q = p^2$, and the law of cosines gives

$$x = \frac{T-2c^2}{2(a-b)}, \qquad y = \frac{T-2a^2}{2(b-c)}, \qquad z = \frac{T-2b^2}{2(c-a)}. \tag{2}$$

Since

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx),$$

the desired identity is equivalent to

$$xy + yz + zx = -\frac{p^2}{2}. \tag{3}$$

Thus it suffices to evaluate the sum of these three pairwise products.

2. Two auxiliary algebraic identities

We shall use the following sums:

$$S_1 = (c-a)(c^2+a^2) + (a-b)(a^2+b^2) + (b-c)(b^2+c^2),$$
$$S_2 = (c-a)c^2a^2 + (a-b)a^2b^2 + (b-c)b^2c^2.$$

We claim that

$$S_1 = D, \qquad S_2 = -qD. \tag{4}$$

The first identity. Expanding the products, the pure cubic terms cancel, so

$$\begin{aligned} S_1 &= a^2c - ac^2 + ab^2 - a^2b + bc^2 - b^2c \\ &= (c-a)\bigl[-ac + b(a+c) - b^2\bigr] \\ &= (c-a)(a-b)(b-c) = D. \end{aligned}$$

In the last step, we used $(a-b)(b-c) = ab - ac - b^2 + bc$.

The second identity. First, factor b^2 out of the second and third terms:

$$\begin{aligned} S_2 &= (c-a)a^2c^2 + b^2\bigl[(a-b)a^2 + (b-c)c^2\bigr] \\ &= (c-a)a^2c^2 + b^2\bigl[a^3 - c^3 - b(a^2-c^2)\bigr] \\ &= (c-a)\bigl[a^2c^2 - b^2(a^2+ac+c^2) + b^3(a+c)\bigr]. \end{aligned}$$

Here we used the factorizations $a^3 - c^3 = (a-c)(a^2+ac+c^2)$ and $a^2 - c^2 = (a-c)(a+c)$. The expression in square brackets can be factored as follows:

$$\begin{aligned} &(a-b)(c-b)(ab+bc+ca) \\ &\quad = \bigl[ac - b(a+c) + b^2\bigr]\bigl[ac + b(a+c)\bigr] \\ &\quad = a^2c^2 - b^2(a+c)^2 + ab^2c + b^3(a+c) \\ &\quad = a^2c^2 - b^2(a^2+ac+c^2) + b^3(a+c). \end{aligned}$$

Consequently,

$$S_2 = (c-a)(a-b)(c-b)q = -(a-b)(b-c)(c-a)q = -qD.$$

This proves both auxiliary identities.

3. The sum of the pairwise products

Using the expressions in (2), we put the pairwise products over the common denominator $4D$. This gives

$$4D(xy + yz + zx) = N,$$

where

$$\begin{aligned} N = {} & (c-a)(T-2c^2)(T-2a^2) \\ & + (a-b)(T-2a^2)(T-2b^2) \\ & + (b-c)(T-2b^2)(T-2c^2). \end{aligned}$$

For example, the denominator of xy is $4(a-b)(b-c)$, so its numerator must be multiplied by $c-a$; the other two terms are treated in the same way.

Expanding the products in the numerator, we group the terms according to the powers of T:

$$\begin{aligned} N = {} & T^2\big[(c-a) + (a-b) + (b-c)\big] \\ & - 2T\big[(c-a)(c^2+a^2) + (a-b)(a^2+b^2) + (b-c)(b^2+c^2)\big] \\ & + 4\big[(c-a)c^2a^2 + (a-b)a^2b^2 + (b-c)b^2c^2\big]. \end{aligned}$$

The first line vanishes because $(c-a)+(a-b)+(b-c) = 0$. The bracketed expressions in the other two lines are precisely S_1 and S_2. Thus, by (4),

$$N = -2TS_1 + 4S_2 = -2TD - 4qD = -2(T+2q)D = -2p^2D.$$

Since $D \neq 0$, we may divide by $4D$ to obtain the desired identity (3):

$$\boxed{xy + yz + zx = -\frac{p^2}{2}.}$$

4. Completing the proof

Substituting this result into the formula for the sum of squares yields

$$x^2 + y^2 + z^2 = (x+y+z)^2 - 2(xy+yz+zx) = (x+y+z)^2 + p^2.$$

Restoring the original meanings of x, y, z and p, we obtain exactly (1). This completes the proof. $\square$

Remark. The proof requires no assumption about the ordering of the side lengths or the signs of the cosines. The identity therefore holds for acute, right, and obtuse triangles alike, provided that their side lengths are pairwise distinct. If two side lengths are equal, at least one denominator in the original expression is zero, so the identity as stated is not defined in that case.