Proposed Problem

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Definitions

The Fibonacci sequence $(F_n)_{n\geq 0}$ and the Lucas sequence $(L_n)_{n\geq 0}$ are defined by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n$$

and

$$L_0 = 2, \qquad L_1 = 1, \qquad L_{n+2} = L_{n+1} + L_n,$$

respectively. Throughout, $n, x, y \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Problem

Proposed Problem. Prove that none of the following equations has a solution in positive integers n, x, y:

$$F_{n+1}^x = 2 + 4F_n^y, \tag{1}$$
$$L_{n+1}^x = 2 + 4L_n^y, \tag{2}$$
$$F_{n+2}^x = F_n + F_{n+7}F_n^y, \tag{3}$$
$$L_{n+2}^x = L_n + L_{n+5}L_n^y. \tag{4}$$

Complete solution

We first establish two elementary facts that will be used repeatedly.

First fact: powers modulo 4

If a is an integer and $k \geq 2$, then

$$a^k \not\equiv 2 \pmod 4.$$

Indeed, if a is odd, then a^k is odd and therefore cannot be congruent to 2 modulo 4. If a is even, write $a = 2b$. Since $k \geq 2$, the number $a^k = 2^k b^k$ is divisible by 4, so $a^k \equiv 0 \pmod 4$.

Second fact: coprimality at distance two

For every $n \geq 1$,

$$\gcd(F_n, F_{n+2}) = 1 \qquad \text{and} \qquad \gcd(L_n, L_{n+2}) = 1.$$

For the Fibonacci sequence, the recurrence and the Euclidean algorithm give

$$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+1} - F_n) = \gcd(F_n, F_{n-1}).$$

Repeated reduction yields

$$\gcd(F_n, F_{n+1}) = \gcd(F_1, F_0) = 1.$$

Consequently,

$$\gcd(F_n, F_{n+2}) = \gcd(F_n, F_{n+1} + F_n) = \gcd(F_n, F_{n+1}) = 1.$$

The same argument for the Lucas sequence gives

$$\gcd(L_n, L_{n+1}) = \gcd(L_0, L_1) = \gcd(2, 1) = 1$$

and hence

$$\gcd(L_n, L_{n+2}) = \gcd(L_n, L_{n+1} + L_n) = \gcd(L_n, L_{n+1}) = 1.$$

We shall also use the following immediate consequence. If a, b, k are positive integers, $\gcd(a, b) = 1$, and $a \mid b^k$, then $a = 1$. Otherwise, a prime divisor p of a would divide b^k, hence $p \mid b$, contrary to $\gcd(a, b) = 1$.

Equation (1)

Assume, for a contradiction, that (1) holds.

First suppose that $x = 1$. Since $F_n \geq 1$ and $y \geq 1$, we have $F_n^y \geq F_n$. Furthermore,

$$F_{n+1} = F_n + F_{n-1} \leq 2F_n < 4F_n$$

for every $n \geq 1$. Therefore

$$F_{n+1} < 4F_n < 2 + 4F_n^y,$$

contradicting (1).

Now suppose that $x \geq 2$. Reducing (1) modulo 4 gives

$$F_{n+1}^x \equiv 2 \pmod 4,$$

which is impossible by the first fact. Thus equation (1) has no solution in positive integers.

Equation (2)

Assume that (2) holds. If $x = 1$, then $L_n^y \geq L_n$. We also have

$$L_{n+1} < 4L_n$$

for every $n \geq 1$. For $n = 1$, this is $L_2 = 3 < 4L_1 = 4$. For $n \geq 2$, the recurrence and the inequality $L_{n-1} < L_n$ give

$$L_{n+1} = L_n + L_{n-1} < 2L_n < 4L_n.$$

It follows that

$$L_{n+1} < 4L_n < 2 + 4L_n^y,$$

contradicting (2).

If $x \geq 2$, reduction modulo 4 gives

$$L_{n+1}^x \equiv 2 \pmod{4},$$

which is again impossible by the first fact. Hence equation (2) has no positive integer solution.

Equation (3)

Suppose that (3) has a positive integer solution. Since $y \geq 1$, its right-hand side factors as

$$F_n + F_{n+7}F_n^y = F_n(1 + F_{n+7}F_n^{y-1}).$$

Therefore

$$F_n \mid F_{n+2}^x.$$

Since $\gcd(F_n, F_{n+2}) = 1$, the second fact implies that $F_n = 1$. The only positive indices for which $F_n = 1$ are $n = 1$ and $n = 2$.

If $n = 1$, then $F_1 = 1$, $F_3 = 2$, and $F_8 = 21$, so (3) becomes

$$2^x = 1 + 21 = 22,$$

which is impossible because 22 is not a power of 2.

If $n = 2$, then $F_2 = 1$, $F_4 = 3$, and $F_9 = 34$, so the equation becomes

$$3^x = 1 + 34 = 35.$$

This is impossible because the left-hand side is divisible by 3, whereas 35 is not. Consequently, equation (3) has no positive integer solution.

Equation (4)

Finally, suppose that (4) holds. Its right-hand side can be factored as

$$L_n + L_{n+5}L_n^y = L_n(1 + L_{n+5}L_n^{y-1}).$$

Thus

$$L_n \mid L_{n+2}^x.$$

Because $\gcd(L_n, L_{n+2}) = 1$, we must have $L_n = 1$. Among positive indices this occurs only for $n = 1$, because

$$L_1 = 1, \qquad L_2 = 3,$$

and the positive recurrence makes the sequence strictly increasing from L_1 onward.

For $n = 1$, we have $L_3 = 4$ and $L_6 = 18$. Equation (4) would therefore give

$$4^x = 1 + 18 = 19,$$

which is impossible. Thus equation (4) has no positive integer solution.

We have proved that none of equations (1)–(4) has a solution in positive integers n, x, y.