Proposed Problem

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Problem

Solve over the rational numbers the system

$$\begin{cases} ab^3 + bc^3 + ca^3 = -59, \\ \dfrac{a+2b}{b+3c} + \dfrac{b+2c}{c+3a} + \dfrac{c+2a}{a+3b} = -\dfrac{16}{21}, \\ \dfrac{a^3}{b+2c} + \dfrac{b^3}{c+2a} + \dfrac{c^3}{a+2b} = -\dfrac{207}{20}. \end{cases} \tag{1}$$

As usual, every denominator occurring in the system is required to be nonzero.

Solution

We shall prove that the complete solution set is

$$\boxed{\begin{aligned}(a,b,c) \in \{&(1,-2,3),\ (-2,3,1),\ (3,1,-2),\\ &(-1,2,-3),\ (2,-3,-1),\ (-3,-1,2)\}.\end{aligned}} \tag{2}$$

1. Reduction to four cyclic invariants

Put

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad r = abc, \qquad \delta = (a-b)(b-c)(c-a). \tag{3}$$

Direct expansion gives

$$ab^3 + bc^3 + ca^3 = \frac{p^2q - 2q^2 - pr + p\delta}{2}. \tag{4}$$

For completeness, the sum of the expression on the left of (4) and its reverse cyclic version is

$$p^2q - 2q^2 - pr,$$

while their difference is $p\delta$, which proves (4).

For the second equation of (1), let D_3 be the product of its three denominators and N_3 the numerator obtained after bringing the three fractions to this common denominator. Expansion in terms of (3) gives

$$\begin{aligned} D_3 &= (b+3c)(c+3a)(a+3b) = 6pq + 10r + 3\delta, \\ N_3 &= 3p^3 + 11pq - 36r - \delta. \end{aligned} \tag{5}$$

Similarly, for the last equation of (1),

$$\begin{aligned} D_2 &= (b+2c)(c+2a)(a+2b) = 3pq + \delta, \\ N_2 &= \frac{1}{2}\big(4p^5 - 15p^3q + 5pq^2 + 19p^2r - 3qr - 3p^2\delta + 3q\delta\big). \end{aligned} \tag{6}$$

Equations (4)–(6) transform (1) into the following three polynomial relations:

$$p^2q - 2q^2 - pr + p\delta + 118 = 0, \tag{7}$$

$$63p^3 + 327pq - 596r + 27\delta = 0, \tag{8}$$

$$40p^5 - 150p^3q + 50pq^2 + 190p^2r - 30qr - 30p^2\delta + 30q\delta + 621pq + 207\delta = 0. \tag{9}$$

The discriminant identity for the cubic with roots a, b, c is

$$\delta^2 = p^2q^2 - 4q^3 - 4p^3r - 27r^2 + 18pqr. \tag{10}$$

If $p = 0$, equation (7) gives $q^2 = 59$, which is impossible for rational q. Hence

$$p \neq 0. \tag{11}$$

2. Elimination of the remaining invariants

Define

$$u = p^2, \qquad X = pr, \qquad Y = p\delta. \tag{12}$$

Equations (7) and (8), the latter multiplied by p, are linear in X, Y. Solving them yields

$$\begin{aligned} X &= \frac{63u^2 + 300uq + 54q^2 - 3186}{569}, \\ Y &= \frac{63u^2 - 269uq + 1192q^2 - 70328}{569}. \end{aligned} \tag{13}$$

Multiply (9) by p, use (12), and then substitute (13). After clearing the nonzero constant denominator, we obtain

$$\begin{aligned} \mathcal{C}(u,q) = 34140q^3 - 14120q^2u + 246744q^2 - 20280qu^2 + 297666qu - 2014260q \\ + 32840u^3 + 13041u^2 + 1504500u - 14557896 = 0. \end{aligned} \tag{14}$$

Multiplying (10) by p^2, and again using (12)–(13), gives

$$\mathcal{D}(u,q) = 374899q^4 + 243870q^3u - 109299q^2u^2 - 44238082q^2$$

$$+\,256065qu^3 + 4713569qu + 63630u^4 - 6737859u^2 + 1305023419 = 0. \tag{15}$$

We now eliminate u. The resultant of the two explicit polynomials (14) and (15), obtained by exact Euclidean elimination, is

$$\mathrm{Res}_u(\mathcal{C}, \mathcal{D}) = -1048211851210\,(q+5)(q^2-59)H(q), \tag{16}$$

where the primitive integer polynomial H is

$$\begin{aligned} H(q) = {} & 299099758355929599680000q^9 - 753563512439583123865600q^8 \\ & + 9392844306586763714836560q^7 - 11278730959274802937812726 0q^6 \\ & + 912852305574174633527039 13q^5 + 178609340748352073336471077 35q^4 \\ & - 783491916947978367871938122 58q^3 - 779258916022765647279346363530q^2 \\ & + 230919127113595325528981123988 7q + 163790244975430802236308631702 45. \end{aligned}$$

We include a short exact certificate that H has no rational root. Modulo 19,

$$H(q) \equiv 14q^9 + q^8 + 12q^7 + 15q^6 + 8q^5 + 18q^4 + 11q^3 + 13q^2 + 9q + 17. \tag{17}$$

For $q = 0, 1, \ldots, 18$, respectively, its values modulo 19 are

q	0	1	2	3	4	5	6	7	8	9
$H(q)$	17	4	18	10	18	4	5	2	16	17

and

q	10	11	12	13	14	15	16	17	18
$H(q)$	5	13	14	8	9	13	14	12	10.

Thus H has no zero in $\mathbb{F}_{19}$. Its leading coefficient is congruent to 14 $\pmod{19}$, so it is not divisible by 19. If a reduced fraction m/n were a rational zero of H, the rational-root theorem would give n dividing the leading coefficient; hence $19 \nmid n$, and reduction modulo 19 would produce a zero of (17), a contradiction. Therefore H has no rational zero.

Since q is rational, $q^2 = 59$ is also impossible. It follows from (16) that

$$q = -5. \tag{18}$$

Substitution into (14)–(15) gives

$$\begin{aligned} \mathcal{C}(u,-5) &= (u-4)(32840u^2 + 245801u + 646374), \\ \mathcal{D}(u,-5) &= (u-4)(63630u^3 - 1025805u^2 - 13573554u - 108345811). \end{aligned}$$

The resultant of the two displayed cofactors is the nonzero integer

$$15393186404041209013321657565 0.$$

Consequently, they have no common root, and the common root of $\mathcal{C}(u,-5)$ and $\mathcal{D}(u,-5)$ must be

$$u = 4. \tag{19}$$

Equations (13), (18), and (19) now give

$$X = -12, \qquad Y = -60. \tag{20}$$

Because $u = p^2 = 4$, we have $p = 2$ or $p = -2$. Moreover,

$$r = \frac{X}{p} = -\frac{12}{p}, \qquad \delta = \frac{Y}{p} = -\frac{60}{p}. \tag{21}$$

If $p = 2$, then $q = -5$, $r = -6$, and the three numbers a, b, c are the roots of

$$t^3 - pt^2 + qt - r = t^3 - 2t^2 - 5t + 6 = (t-1)(t+2)(t-3). \tag{22}$$

Thus $\{a, b, c\} = \{1, -2, 3\}$. In this case (21) gives $\delta = -30$, which selects precisely the three cyclic orderings

$$(1, -2, 3), \qquad (-2, 3, 1), \qquad (3, 1, -2). \tag{23}$$

If $p = -2$, all three entries are the negatives of those in (23), giving the remaining three triples in (2).

3. Verification

It is enough to check $(a, b, c) = (1, -2, 3)$, because all three expressions in (1) are invariant under simultaneous sign reversal, and the system is invariant under cyclic permutation. We have

$$ab^3 + bc^3 + ca^3 = -8 - 54 + 3 = -59,$$

$$\frac{-3}{7} + \frac{4}{6} + \frac{5}{-5} = -\frac{3}{7} + \frac{2}{3} - 1 = -\frac{16}{21},$$

and

$$\frac{1}{4} - \frac{8}{5} + \frac{27}{-3} = \frac{1}{4} - \frac{8}{5} - 9 = -\frac{207}{20}.$$

All denominators are nonzero. Hence every triple in (2) is valid, and the preceding elimination proves that there are no others.