Proposed Problem

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Problem

Let $n \geq 3$ be an integer, and let $x_1, x_2, \ldots, x_n$ be positive real numbers satisfying

$$x_1 + x_2 + \cdots + x_n = 1.$$

With cyclic indices, so that $x_{n+1} = x_1$, prove the refined inequality

$$\sum_{i=1}^{n} \frac{1}{x_i} + n^2 \sum_{i=1}^{n} x_i x_{i+1} \geq n^2 + n + \frac{n^2}{2} \sum_{i=1}^{n} \left(x_i + x_{i+1} - \frac{2}{n} \right)^2. \tag{1}$$

In fact, establish the following exact remainder identity:

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{x_i} + n^2 \sum_{i=1}^{n} x_i x_{i+1} \\
&\quad = n^2 + n + n^2 \sum_{i=1}^{n} \frac{1 - x_i}{x_i} \left(x_i - \frac{1}{n} \right)^2 + \frac{n^2}{2} \sum_{i=1}^{n} \left(x_i + x_{i+1} - \frac{2}{n} \right)^2.
\end{aligned} \tag{2}$$

Deduce that

$$\sum_{i=1}^{n} \frac{1}{x_i} + n^2 \sum_{i=1}^{n} x_i x_{i+1} \geq n^2 + n, \tag{3}$$

and determine all equality cases.

Solution

Define

$$y_i = x_i - \frac{1}{n} \qquad (i = 1, 2, \ldots, n).$$

Then

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i - 1 = 0$$

and, by cyclic convention, $y_{n+1} = y_1$.

We first rewrite the reciprocal sum. For each i, direct expansion gives

$$\frac{1}{x_i} = n - n^2\left(x_i - \frac{1}{n}\right) + n^2\frac{\left(x_i - \frac{1}{n}\right)^2}{x_i}.$$

Indeed, after multiplying by x_i, the right-hand side reduces to 1. Summing this identity and using $\sum_{i=1}^{n}(x_i - 1/n) = 0$, we obtain

$$\sum_{i=1}^{n}\frac{1}{x_i} = n^2 + n^2\sum_{i=1}^{n}\frac{y_i^2}{x_i}. \tag{4}$$

Next, since $x_i = 1/n + y_i$, we have

$$\begin{aligned}\sum_{i=1}^{n} x_i x_{i+1} &= \sum_{i=1}^{n}\left(\frac{1}{n} + y_i\right)\left(\frac{1}{n} + y_{i+1}\right)\\ &= \frac{1}{n} + \frac{1}{n}\sum_{i=1}^{n}(y_i + y_{i+1}) + \sum_{i=1}^{n} y_i y_{i+1}\\ &= \frac{1}{n} + \sum_{i=1}^{n} y_i y_{i+1}.\end{aligned}$$

Consequently,

$$n^2\sum_{i=1}^{n} x_i x_{i+1} = n + n^2\sum_{i=1}^{n} y_i y_{i+1}. \tag{5}$$

Adding (4) and (5) yields

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{x_i} + n^2\sum_{i=1}^{n} x_i x_{i+1} - (n^2 + n)\\ &\quad = n^2\left(\sum_{i=1}^{n}\frac{y_i^2}{x_i} + \sum_{i=1}^{n} y_i y_{i+1}\right).\end{aligned} \tag{6}$$

For every i,

$$\frac{y_i^2}{x_i} = \frac{1 - x_i}{x_i}y_i^2 + y_i^2.$$

Moreover, cyclicity gives

$$\begin{aligned}\sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} y_i y_{i+1} &= \frac{1}{2}\sum_{i=1}^{n}\left(y_i^2 + 2y_i y_{i+1} + y_{i+1}^2\right)\\ &= \frac{1}{2}\sum_{i=1}^{n}(y_i + y_{i+1})^2.\end{aligned}$$

Substitution in (6), together with

$$y_i + y_{i+1} = x_i + x_{i+1} - \frac{2}{n},$$

proves the exact identity (2).

Because all the variables are positive and their sum is 1, we have $0 < x_i < 1$ for every i. Hence

$$\frac{1 - x_i}{x_i} > 0,$$

and both sums on the right-hand side of (2) are nonnegative. Dropping only the first of these sums gives the refinement (1); dropping both gives (3).

Finally, equality in either (1) or (3) forces

$$\sum_{i=1}^{n} \frac{1 - x_i}{x_i} \left(x_i - \frac{1}{n} \right)^2 = 0.$$

Every coefficient $(1 - x_i)/x_i$ is strictly positive, so every squared term must vanish. Thus

$$x_1 = x_2 = \cdots = x_n = \frac{1}{n}.$$

Conversely, these values give equality. Therefore the equality case is unique, and the constant $n^2 + n$ in (3) is sharp.