Proposed Problem

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Problem. Let $n \geq 2$ be an integer, let $\lambda > 0$, and let $x_1, x_2, \ldots, x_n$ be positive real numbers satisfying

$$x_1 + x_2 + \cdots + x_n = n.$$

Using the cyclic convention $x_{n+1} = x_1$, prove the sharpened inequality

$$\sum_{i=1}^{n} \frac{x_i^5}{\lambda x_i + x_{i+1} + 1} \geq \frac{1}{n(\lambda+2)} \left(\sum_{i=1}^{n} x_i^{5/2} \right)^2. \tag{1}$$

Deduce the explicit stability refinement

$$\sum_{i=1}^{n} \frac{x_i^5}{\lambda x_i + x_{i+1} + 1} \geq \frac{1}{n(\lambda+2)} \left(n + \frac{3}{2} \sum_{i=1}^{n} (x_i - 1)^2 \right)^2 \geq \frac{n}{\lambda+2}. \tag{2}$$

Determine all equality cases.

Solution. For $i = 1, 2, \ldots, n$, set

$$d_i = \lambda x_i + x_{i+1} + 1.$$

Every d_i is positive. Moreover, cyclically shifting the indices does not change their sum, and hence

$$\begin{aligned} \sum_{i=1}^{n} d_i &= \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_{i+1} + n \\ &= \lambda n + n + n = n(\lambda+2). \end{aligned}$$

Apply the Cauchy–Schwarz inequality in Engel form:

$$\begin{aligned} \sum_{i=1}^{n} \frac{x_i^5}{\lambda x_i + x_{i+1} + 1} &= \sum_{i=1}^{n} \frac{(x_i^{5/2})^2}{d_i} \\ &\geq \frac{\left(\sum_{i=1}^{n} x_i^{5/2} \right)^2}{\sum_{i=1}^{n} d_i} \\ &= \frac{1}{n(\lambda+2)} \left(\sum_{i=1}^{n} x_i^{5/2} \right)^2. \end{aligned}$$

This proves (1).

We next establish a quantitative lower estimate for the power sum. For every $t > 0$, direct factorization gives

$$\begin{aligned} t^{5/2} - 1 - \frac{5}{2}(t-1) - \frac{3}{2}(t-1)^2 &= \frac{t}{2}(\sqrt{t}-1)^2(2\sqrt{t}+1) \\ &\geq 0. \end{aligned}$$

Therefore,

$$t^{5/2} \geq 1 + \frac{5}{2}(t-1) + \frac{3}{2}(t-1)^2,$$

with equality if and only if $t = 1$. Applying this inequality to each x_i and using

$$\sum_{i=1}^{n}(x_i - 1) = \sum_{i=1}^{n} x_i - n = 0,$$

we obtain

$$\sum_{i=1}^{n} x_i^{5/2} \geq n + \frac{3}{2}\sum_{i=1}^{n}(x_i - 1)^2. \tag{3}$$

Both sides of (3) are positive. Thus, substituting its square into (1), we find

$$\begin{aligned} \sum_{i=1}^{n} \frac{x_i^5}{\lambda x_i + x_{i+1} + 1} &\geq \frac{1}{n(\lambda+2)}\left(n + \frac{3}{2}\sum_{i=1}^{n}(x_i-1)^2\right)^2 \\ &= \frac{n}{\lambda+2} + \frac{3}{\lambda+2}\sum_{i=1}^{n}(x_i-1)^2 \\ &\quad + \frac{9}{4n(\lambda+2)}\left(\sum_{i=1}^{n}(x_i-1)^2\right)^2 \\ &\geq \frac{n}{\lambda+2}. \end{aligned}$$

This proves the complete chain (2). In particular, it not only generalizes the original three-variable inequality, but also measures explicitly how much the left-hand side exceeds its constant lower bound when the variables move away from their common value.

It remains to discuss sharpness and equality. If $x_1 = x_2 = \cdots = x_n = 1$, then every denominator equals $\lambda + 2$, so

$$\sum_{i=1}^{n} \frac{x_i^5}{\lambda x_i + x_{i+1} + 1} = \frac{n}{\lambda+2}.$$

Thus all the bounds above are attained. Conversely, equality in (3) forces $x_i = 1$ for every i. Hence equality in the complete refined inequality holds if and only if

$$x_1 = x_2 = \cdots = x_n = 1.$$

Consequently, the constant $n/(\lambda+2)$ is best possible. The factor $1/[n(\lambda+2)]$ in (1) is also best possible, because equality is attained at the same point.

For completeness, the exact remainder in the first Cauchy–Schwarz step is

$$\sum_{i=1}^{n} \frac{x_i^5}{d_i} - \frac{\left(\sum_{i=1}^{n} x_i^{5/2}\right)^2}{n(\lambda+2)} = \frac{1}{n(\lambda+2)} \sum_{1\le i<j\le n} \frac{\left(d_j x_i^{5/2} - d_i x_j^{5/2}\right)^2}{d_i d_j} \ge 0.$$

This identity provides a direct verification of both the refinement and its sharpness.