Proposed Problem

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Let $m \geq 2$ and $q \geq 1$ be integers. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\sum_{j=0}^{m-1} f^{\circ j}(x) = 0 \qquad (x \in \mathbb{R}). \tag{1}$$

If $a, b \in \mathbb{R}$ and $a > |b|$, evaluate

$$\Omega_{m,q}(a,b) = \int_0^\pi f^{\circ mq}\left(\frac{1}{a + b\cos x}\right) dx. \tag{2}$$

Solution. *Under the assumptions above,*

$$\boxed{\Omega_{m,q}(a,b) = \frac{\pi}{\sqrt{a^2 - b^2}}.}$$

No continuity, monotonicity, measurability, or injectivity assumption on f is needed.

Proof. We first determine the iterate forced by the functional equation. Writing out (1) gives

$$x + f(x) + f^{\circ 2}(x) + \cdots + f^{\circ(m-1)}(x) = 0. \tag{3}$$

Because (3) holds for every real input, it also holds with $f(x)$ in place of x. Thus

$$f(x) + f^{\circ 2}(x) + \cdots + f^{\circ(m-1)}(x) + f^{\circ m}(x) = 0. \tag{4}$$

Subtracting (3) from (4), all intermediate terms cancel and yield

$$f^{\circ m}(x) - x = 0.$$

Consequently,

$$f^{\circ m} = \mathrm{id}\,. \tag{5}$$

Iterating (5) q times gives

$$f^{\circ mq} = (f^{\circ m})^{\circ q} = \mathrm{id}\,.$$

Therefore the integrand in (2) simplifies pointwise:

$$f^{\circ mq}\left(\frac{1}{a + b\cos x}\right) = \frac{1}{a + b\cos x}.$$

The condition $a > |b|$ implies $a + b\cos x \geq a - |b| > 0$, so the remaining integral is proper. Use the tangent half-angle substitution

$$t = \tan\frac{x}{2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2\,dt}{1+t^2}.$$

As x goes from 0 to π, t goes from 0 to $+\infty$. Hence

$$\begin{aligned}
\Omega_{m,q}(a,b) &= 2\int_0^\infty \frac{dt}{(a+b)+(a-b)t^2} \\
&= \frac{2}{\sqrt{(a+b)(a-b)}}\left[\arctan\left(t\sqrt{\frac{a-b}{a+b}}\right)\right]_0^\infty \\
&= \frac{2}{\sqrt{a^2-b^2}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{a^2-b^2}}.
\end{aligned}$$

This proves the stated formula. □